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UNIT 1: INTRODUCTION TO OPERATIONS RESEARCH

Structure

1.1 HISTORY AND BACKGROUND OF OPERATIONS RESEARCH

In the books of management one often finds a specific period of the development of management thought, called the *Period of Scientific Management*. It was in 1885 that Fredrick W. Taylor, "father of scientific management", developed the scientific management theories. It was also called the Modern era when rapid development of concepts, theories and techniques of management took place. During World War II, production bottlenecks forced the government of Great Britain to look up to scientists and engineers to help achieve maximum military production. These scientists and engineers created mathematical models to find the solution of the problems about increasing production of military equipment. This branch of study was called *Operations Research* (OR). Since it was used in the research in war operations of armed forces. These problems of the armed forces seemed to be similar to those that occurred in production systems. Because of the success of OR in military operations and approach to war problems it began to be used in industry as well.

1.2 DEFINITION OF OPERATIONS RESEARCH

Many authors have given different interpretation to the meaning of Operations Research as it is not possible to restrict the scope of Operations Research in a few sentences. Students must understand that there is no need to single definition of Operations Research which is acceptable to everyone. Two of the widely accepted definitions are provided below for understanding the concept of Operations Research.

"*Operations Research is concerned with scientifically deciding how best to design and operate man-machine system usually under conditions requiring the allocation of scarce resources.*"

– Operations Research Society of America

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The salient features of the above definition are :

(*a*) It is a scientific decision-making technique.

(*b*) It deals with optimizing (maximizing) the results.

(*c*) It is concerned with man-machine systems.

(*d*) The resources are limited.

"*Operations Research is a scientific approach to problem solving for executive management.*"

– HM Wagner

The above definition lays emphasis on :

(*a*) OR being a scientific technique.

(*b*) It is a problem-solving technique.

(*c*) It is for the use of executives who have to take decisions for the organizations.

A close observation of the essential aspects of the above two definitions will make it clear that both are in reality conveying the same meaning. Other definitions of OR also converge on these essential features. One need not remember the definitions word by word but understand the true meaning of the definition provided by different authors. The emphasis has to be on the application of technique so that organizations are benefitted. Hence, the real work of any managerial technique is the ability of the organizations to take advantage for meeting their objectives.

1.3 OPERATIONS MANAGEMENT, PRODUCTION MANAGEMENT, SYSTEM MANAGEMENT AND OPERATIONS RESEARCH

All the above subjects are inter-related and one must understand the fundamental concepts of these subjects before one is ready to study the details.

Any production function brings together men, machines and materials. These are used to provide goods and services, which satisfy the needs, wants and desires of the people. For long, the term 'production' has been associated with factory like situations where goods are produced in the physical sense. In fact, a factory is defined as "*a premises where people are employed for making, altering, repairing, ornamenting, finishing, cleaning, working, breaking, demolishing or adopting any article for sale.*" For example, in a factory the mass production of any household product or goods may take place. The management of production of such goods is important but equally important is the management of the service part associated with it. Similarly, one can distinguish between the production of say McDonald's burger which is a product and its delivery, which is a service. If we generalize the concept of production as the 'process through which goods and services are created', both manufacturing of goods and service organization can be included in production management. Thus, non-manufacturing processes like health, transport, banks, education etc., come under the scope of production management. It is because of this reason that term 'production' or 'operations management' has been suggested by many authors to include the application of techniques of management of men, machines and materials.

So, general concept of 'Operations' and not production can include both manufacturing and service organizations. There is an operation function involved in all enterprises, big or small.

1.4 SALIENT FEATURES OF OPERATIONS RESEARCH

After having understood the basic concept of OR and the need, one can easily understand its salient features.

1. **System Approach :** OR is a systematic approach as is clear from the conceptual model of OR explained above. It encompasses all the sub-systems and departments of an organization. Since it is a technique that effects the entire organization, optimizing results of one part of the organization is not the proper use of OR. Before applying OR techniques the management must understand its impact and implications on the entire organization.
2. **OR is both a Science and an Art :** OR has the scientific orientation because of its inherent methodology and scientific methods are used for problem-solving. But its implementation needs the art of taking the entire organization along. OR does not perform experiment but helps in finding out solutions. OR must take into account the human factor which is the most important factor in implementing any technique/ methods of problem-solving.
3. **Interdependency Approach :** Problem of organizations could be related with economics, engineering, infrastructure related with markets, management of human resources and so on. If OR has to find a solution to problems related to diverse fields, the OR team must be constituted of members with background disciplines of science, management and engineering etc. Only then, practical solutions which can be implemented, can be found to the advantage of organizations.
4. **Management Decision-making :** Management of any organization has to make decision, which has, impact on its profitability. All business organizations exist to make profits. Non-business organizations like hospitals, educational institutions, NGOs etc., generate profits by reducing the inputs and increasing the outputs through effective and efficient management. Decision-making involves generating different alternatives and selecting the best under the given situation. OR helps in making the right decisions.
5. **Quantitative Technique :** OR is a quantitative technique, which uses mathematical models and finds rational quantitative solutions to the managerial problems. The management may use the OR inputs and take into account the quantitative analysis of the problem in finding the solution in the best interest of the organization.
6. **Use of Information Technology (IT) :** OR extensively uses the IT for complex mathematical problems to its advantage. OR approach to decision-making depends heavily on the use of computers.

1.5 TOOLS OF OPERATION RESEARCH

Operation Research is a very versatile science and has many tools/techniques, which can be used for problem solving. However, it is not possible to list all these techniques as everyday new methods in the use of OR are being developed. Some of the tools of OR are discussed in the succeeding paragraphs :

1. **Linear Programming (LP) :** Most of the industrial and business organisations have the objectives of minimizing costs and maximizing the profits. LP deals with maximizing a given objective. Since the objective function and boundry conditions are linear in nature, this mathematical model is called *Linear Programming Model*. It is a mathematical technique used to allocate limited resources amongst competing demands in an optimal

manner. The application of LP requires that there must be a well-defined objective function (like maximizing profits and minimizing costs) and there must be constraints on the amount and extent of resources available for satisfying the objective function.

2. **Queuing Theory :** In real life situations, the phenomenon of waiting is involved whether it is the people waiting to buy goods in a shop, patients waiting outside an Out Patient Department (OPD), vehicles waiting to be serviced in a garage and so on. Because in general, customer's arrival and his service time is not known in advance; hence a queue is formed. Queuing or waiting line theory aims at minimizing the overall cost due to servicing and waiting. How many servicing facilities can be added at what cost to minimize the time in queue is the aim in the application of this theory.
3. **Network Analysis Technique :** A network can be used to present or depict the activities necessary to complete a project. This helps us in planning, scheduling, monitoring and control of large and complex projects. The project may be developing a new battle tank, construction of dam or a space flight. The project managers are interested in knowing the total project completion time, probability that a project can be completed by a particular time, and the least cost method of reducing the total project completion time. Techniques like Programme Evaluation and Reviewing Technique (PERT) and Critical Path Method (CPM) are part of network analysis. These are popular techniques and widely used in project management.
4. **Replacement Theory Model :** All plants, machinery and equipment needs to be replaced at some point of time, either because there is deterioration in their efficiency or because new and better equipment is available and the old one has become obsolete. Sooner or later the equipment needs to be replaced. The decision to be taken by the management involves consideration of the cost of new equipment which is to be purchased and what can be recovered from the old equipment through its sale, or its scrap value, the residual life of the old equipment and many other related aspects. These are important decisions involving investment of capital and need to be taken very carefully.
5. **Inventory Control :** Inventory includes all the stocks of material, which an organization buys for production/manufacture of goods and services for sale. It will include raw material; semi-finished and finished products, spare parts of machines, etc. Managers face the problems of how much of raw material should be purchased, when should it be purchased and how much should be kept in stock. Overstocking will result in locked capital not available for other purposes, whereas under-stocking will mean stock-out and idle manpower and machine resulting in reduced output. It is desirable to have just the right amount of inventory at the right time. Inventory control models can help us in finding out the optimal order size, reorder level, etc., so that the capital resources are conserved and maximum output ensured.
6. **Integer Programming :** Integer programming deals with certain situations in which the variable assumes non-negative integer (complete or whole number) values only. In LP models the variable may take even a fraction value and the figures are rounded off to the nearest integer to get the solution, *i.e.*, number of vehicles available in a problem cannot be in fractions. When such rounding off is done the solution does not remain an optimal solution. In integer programming the solution containing unacceptable and fractional values are ruled out and the next best solution using whole numbers is obtained. An integer programming may be called *mixed* or *pure depending* on whether some or all the variables are restricted to integer values.
7. **Transportation Problems :** Transportation problems are basically LP model problems. This model deals with finding out the minimum transportation cost for transporting

the single commodity from a number of sources to number of destinations. Typical problem involves transportation of some manufactured products (say cars in 3 different plants) and these have to be sent to the warehouses of various dealers in different parts of country. This may be understood as a special case of simplex method developed for LP problems, allocating scare resources to competing demands. The main purpose of the transportation is to schedule the dispatch of the single product from different sources like factories to different destinations as total transportation cost is minimized.

8. **Decision Theory and Games Theory :** Information for making decisions is the most important factor. Many models of OR assume availability of perfect information which is called *decision-making under certainty*. However, in real life situations, only partial or imperfect information is available. In such a situation we have two cases, either decision under risk or decision under uncertainty. Hence from the point of view of availability of information, there are three cases, certainty and uncertainty, the two extreme cases and risk is the "in-between" case.

   Games theory is concerned with decision-making in a conflict situation where two or more intelligent opponents try to optimize their own decision. In Games theory, an opponent is referred to as a player and each player has a number of choices. The Games theory helps the decision maker to analyse the course of action available to his opponent. In decision theory, we use decision tree which can be graphically represented to solve the decision-making problems.

9. **Assignment Problems :** We have the problem of assigning a number of tasks to a number of persons who may use machines. The objective is to assign the jobs to the machines in such a way that the cost is least. This may be considered a special case of LP transportation model. Here jobs may be treated as 'services' and machines may be considered the 'destinations'. Assignment of a particular job to a particular person so that all the jobs can be completed in shortest possible time hence incurring the least cost, is the assignment problem.

10. **Markov Analysis :** Markov analysis is used to predict future conditions. It assumes that the occurrence of a future state depends upon the immediately preceding state and only on it. It is based on the probability theory and predicts the change in a system over a period of time if the present behaviour of the system is known. Predicting market share of the companies in future as also whether a machine will function properly or not in future, are examples of Markov analysis.

11. **Simulation Techniques :** Since all real life situations cannot be represented mathematically, certain assumptions are made and dynamic models which act like the real processes are developed. It is very difficult to develop simulation models which can give accurate solutions to the problems, but this is a good method of problem solving, when the problems are very complex and cannot be solved otherwise.

1.6 IMPORTANT APPLICATIONS OF OPERATION RESEARCH

In today's world where decision-making does not depend on intuition, managerial techniques are widely used. All the applications of OR cannot be listed because OR as a tool finds new application everyday. It finds typical applications in many activities related to work planning.

Some important applications of OR are :

1. **Manufacturing/Production**
   - Production planning and control
   - Inventory management.

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2. **Facilities Planning**
   - Design of logistic systems
   - Factory/building location and size decisions
   - Transportation, loading and unloading
   - Planning warehouse locations.
3. **Accounting**
   - Credit policy decisions
   - Cash flow and fund flow planning.
4. **Construction Management**
   - Allocation of resources to different projects in hand
   - Workforce/labour planning
   - Project management (scheduling, monitoring and control).
5. **Financial Management**
   - Investment decisions
   - Portfolio management.
6. **Marketing Management**
   - Product-mix decisions
   - Advertisement/Promotion budget decisions
   - Launching new product decisions.
7. **Purchasing Decisions**

   Inventory management (optimal level of purchase), Optimal re-ordering.
8. **Personnel Management**
   - Recruitment and selection of employees
   - Designing training and development programmes
   - Human Resources Planning (HRP).
9. **Research and Development**
   - Planning and control of new research and development projects.
   - Product launch planning.

1.7 PITFALLS IN THE USE OF OPERATION RESEARCH FOR DECISION-MAKING

The first stage of OR application after collecting data/information through observation is the formulation of the problem. It is the most important and most difficult task in OR application. Have the OR team been able to identify the right problem for finding the solution ? Has the problem been accurately defined in unambiguous manner ? Selecting and developing a suitable model is not an easy task. The model must represent the real life situation as far as possible. Collection of data needs a lot of time by a number of people. It is time-consuming and expensive process. Collection of data is done either by observation or from the previous recorded data. When a system is being observed by the OR team, it effects the behaviour of the persons performing the task. The very fact that the workers know that they are being observed is likely to change their work behaviour. The second method of data collection, the records, are never reliable and do not provide sufficient information which is required.

As OR problem-solving techniques is very time-consuming, the quality of decision-making may become a causality. The management has to make a decision either way. Decision based on insufficient or incomplete information will not be the best decision. A reasonably good solution without the use of OR may be preferred by the management as compared to a slightly better solution provided by the use of OR which is very expensive in time and money.

Due to the above reasons, many OR specialists try and fit the solution they have, to the problem. This is dangerous and unethical and organizations must guard against this.

1.8 LIMITATIONS OF OPERATIONS RESEARCH

Operation Research is an extremely powerful tool in the hands of a decision-maker and to that extent the advantage of OR techniques are immense. Some of them are :

(*a*) *It helps in optimum use of resources.* LP techniques suggest many methods of most effective and efficient ways of optimally using the production factors.

(*b*) *Quality of decision can be improved by suitable use of OR* techniques. If a mathematical model representing the real life situation is well-formulated representing the real life situation, the computation tables give a clear picture of the happenings (changes in the various elements i.e., variables) in the model. The decision-maker can use it to his advantage, specially if computerised software can be used to make changes in variables as per requirement.

The limitations of OR emerge only out of the time and cost involved as also the problem of formulating a suitable mathematical model, otherwise, as suggested above, it is a very powerful medium of getting the best out of limited resources. So, the problem is its application rather than its utility, which is beyond doubt. Some of the limitations are :

(*a*) *Large number of cumbersome computations*. Formulation of mathematical models which takes into account all possible factors which define a real life problem is difficult. Because of this, the computations involved in developing relationships in very large variables need the help of computers. This discourages small companies and other organisations from getting the best out of OR techniques.

(*b*) *Quantification of problems*. All the problems cannot be qualified properly as there are a large number of intangible factors, such as human emotions, human relationship and so on. If these intangible elements/variables are excluded from the problem even though they may be more important than the tangible ones, the best solution cannot be determined.

(*c*) *Difficult to conceptualize and use by the managers.* OR applications is a specialist's job, these persons may be mathematicians or statisticians who understand the formulation of models, finding solution and recommending the implementation. The managers really do not have the hang of it. Those who recommend a particular OR technique may not understand the problem well enough and those who have to use may not understand the 'why' of that recommendation. This creates a 'gap' between the two and the results may not be optimal.

1.9 TIPS ON FORMULATING LINEAR PROGRAMMING MODELS

(*a*) *Read the statement of the problem carefully.*

(*b*) *Identify the decision variables.* These are the decisions that are to be made. What set of variables has a direct impact on the level of achievement of the objectives and can be controlled by the decision-maker ? Once these variables are identified, list them providing a written definition (*e.g.*, x_1 = number of units produced and sold per week of product 1, x_2 = number of units produced and sold per week of product 2).

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(*c*) *Identify the objective.* What is to be maximised or minimised ? (*e.g.*, maximize total weekly profit from producing product 1 and 2).

(*d*) *Identify the constraints.* What conditions must be satisfied when we assign values to the decision variables? You may like to write a verbal description of the restriction before writing the mathematical representation (*e.g.*, total production of product 1 > 100 units).

(*e*) *Write out the mathematical model.* Depending on the problem, you might start by defining the objective function on the constraints. Do noorget to include then-negativity constraints.

1.10 GRAPHICAL SOLUTION

Example 1.1: *A firm manufactures two products. The products must be processed through one department. Product A requires 4 hours per unit and product B requires 2 hours per unit. Total production time available for the coming week is 60 hours. A restriction in planning the production schedule, therefore, is the total hours used in producing the two products cannot exceed. Also, since each variable represents a production quantity, neither variable can be negative. Determine the combination of products A and B that can be produced ?*

Solution. Let x_1 represents the number of units produced of product A and x_2 represents the number of units produced of B. Then the restriction is represented by

$$4\,x_1 + 2\,x_2 \leq 60$$

The problem also implies that $x_1 \geq 0$ and $x_2 \geq 0$

In equation $4x_1 + 2\,x_2 = 60$

We can put different values of one variable to get the value of the other variable *i.e.*, $x_1 = 0$, $x_2 = 30$ and $x_2 = 0$, $x_1 = 15$. Hence point A is ($x_1 = 0$, $x_2 = 15$) and Point B is($x_1 = 15$, $x_2 = 0$). This is shown graphically here.

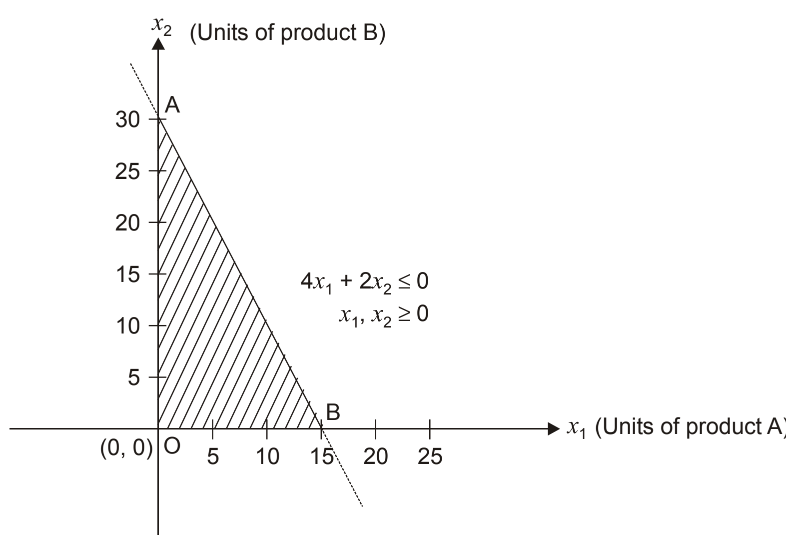

Fig. 1.1

The shaded area represents the comation of products A and B which cbe produced.

1.11 SUMMARY

- The development of management thought, called the *Period of Scientific Management*. It was in 1885 that Fredrick W. Taylor, "father of scientific management", developed the scientific management theories
- **Linear Programming (LP) :** Most of the industrial and business organisations have the objectives of minimizing costs and maximizing the profits. LP deals with maximizing a given objective. Since the objective function and boundry conditions are linear in nature, this mathematical model is called *Linear Programming Model.*
- Queuing or waiting line theory aims at minimizing the overall cost due to servicing and waiting. How many servicing facilities can be added at what cost to minimize the time in queue is the aim in the application of this theory.
- A network can be used to present or depict the activities necessary to complete a project. This helps us in planning, scheduling, monitoring and control of large and complex projects.
- The decision to be taken by the management involves consideration of the cost of new equipment which is to be purchased and what can be recovered from the old equipment through its sale, or its scrap value, the residual life of the old equipment and many other related aspects
- Inventory includes all the stocks of material, which an organization buys for production/ manufacture of goods and services for sale.
- Integer programming deals with certain situations in which the variable assumes non-negative integer (complete or whole number) values only.
- Transportation problems are basically LP model problems. This model deals with finding out the minimum transportation cost for transporting the single commodity from a number of sources to number of destinations
- Information for making decisions is the most important factor. Many models of OR assume availability of perfect information which is called *decision-making under certainty*
- We have the problem of assigning a number of tasks to a number of persons who may use machines. The objective is to assign the jobs to the machines in such a way that the cost is least. This may be considered a special case of LP transportation model
- The first stage of OR application after collecting data/information through observation is the formulation of the problem. It is the most important and most difficult task in OR application

1.12 REVIEW AND DISCUSSION QUESTIONS

1. What is the concept of Operation Research ? Write a detailed note on its development.
2. Discuss significance and scope of OR in business and industry.
3. What are the different phases of OR ? How is OR helpful in decision-making ?
4. Discuss briefly various steps involved in solving an OR problem. Illustrate with one example from the functional area of your choice.
5. Explain applications of Operations Research in business.

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6. What is the significance and scope of Operation Research in the development of Indian Economy ?

7. What is the role of OR in modern day business ? Give examples in support of your answer.

8. Discuss the meaning, significance and scope of Operations Research. Describe some methods of OR.

9. Illustrate and explain various features of OR.

10. Define Operations Research in your own words and explain various tools of OR.

11. Give the role and significance of OR in business and industry for scientific decision-making.

12. "Operations Research is an aid for the executive in making his decisions by providing him with needed quantitative information based on the scientific method of analysis." Discuss the statement and give examples to illustrate how OR is helpful in decision-making.

13. Briefly explain the technique of OR and its uses in India. Which of the three techniques is most widely used in India and Why ?

14. "OR is useful only if applied with Information Technology." Comment.

15. Many believe that OR is a technique which helps in resolving conflicts between production, finance, marketing and personnel functions of a manufacturing unit. Do you agree ? Explain your answer giving examples.

16. Define OR and discuss its scope.

17. Discuss the significance and scope of Operations Research in modern management.

18. Write a detailed note on the use of models for decision-making. Your answer should specifically cover the following :

(i) Need for model building

(ii) Type of model appropriate to the situation

(iii) Steps involved in the construction of a model

(iv) Setting up criteria for evaluating different alternatives

(v) Role of random numbers.

19. Comment on the following statements :

(*a*) (*i*) OR is the art of winning wars without actually fighting them.

(*ii*) OR is the art of finding bad answers where worse exist.

(*b*) OR is not more than a quantitative analysis of the problems.

(*c*) OR advocates a system approach and is concerned with optimisation. It provides a quantitative analysis for decision-making.

(*d*) OR replaces management by personality.

20. Suggest a suitable OR model, giving reasons if any, for each of the following OR problems:

(a) Stockpiling of crackers prior to Diwali.

(b) Decision to replace the fleet of buses of a transport corporation after use for 12 years even though some of them may be in working condition.

(c) A book vendor deciding to place order for books before a new school session begins.

(d) Modifications in the design of a new product due for launch in near future.

(e) Statistical forecasting for sale of ice-cream.

21. ‘‘OR replaces management by personality.’’ Discuss.

22. What are the steps involved in OR problems?

23. What are the different types of models used in OR ? Explain in detail.

24. What are the situations when OR techniques will be applicable ?

UNIT 2: DECISION THEORY

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Structure

2.1 INTRODUCTION

Management has to make decisions. We have deal with certain situations where we have perfect information; such decisions are made under certainty. Most of the managers make major financial investments and other decisions related with production, marketing, etc., with less than complete information. Decision theory, provides a rational approach in dealing with such situations, where the information is incomplete and uncertain about future conditions. The management must make decisions under such circumstances. With the help of decision theory best possible decision under a particular situation can be taken.

In decision theory, a number of statistical techniques can help the management in making rational decisions. One such statistical decision theory is known as Bayesin decision theory.

2.2 DECISION THEORY APPROACH

While discussing the approach, it will be helpful for us to take a real life situation and relate it with the steps involved in taking decisions. Let us take the case of a manufacturing company, which is interested in increasing its production to meet the increasing market demand.

Step I. Determine all possible alternatives

The first obvious step involved before making a rational decision is to list all the viable alternatives available in a particular situation. In the example considered above, the following options are available to the manufacturer :

(*a*) Expand the existing manufacturing facilities (Expansion);

(*b*) Setup a new plant (New facilities);

(*c*) Engage other manufacturers to produce for him as much as is the extra demand (Sub contracting).

Step II. Identify the future scenario

It is very difficult to identify the exact events that may occur in future. However, it is possible to list all that can happen. The future events are not under the control of the decision-maker. In decision theory, identifying the future events is called the *state of nature*. In the case which we have taken of a particular manufacturing company, we can identify the following future events :

(*a*) Demand continues to increase (High demand) (*b*) Moderate demand

(*c*) Demand starts coming down (Low demand)

(*d*) The product does not remain in demand (No demand).

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Step III. Preparing a payoff table

The decision-maker has to now find out possible payoffs, in terms of profits, if any, of the possible events taking place in future. Putting all the alternatives together (Step I) in relation to the state of nature (Step II) gives us the payoff table. Let us prepare the payoff table for our manufacturig company.

Alternatives	**State of nature**			
	High Demand	**Moderate Demand**	**Low Demand**	**No Demand**
Expansion	1	2	3	4
Add New Facilities	5	6	7	8z
Sub-Contact	9	10	11	12

If expansion is carried out and the demand continues to be high (one of the 12 alternatives), the payoff is going to be maximum in terms of profit of say Rs. X. However, if expansion is carried out and there is no demand (situation 4), the company will suffer a loss.

Step IV. Select the best alternative

The decision-maker will, of course, select the best course of action in terms of payoff. However, it must be understood that the decision may not be based on purely quantitative payoff in terms of profit alone, the decision-maker may consider other qualitative aspects like the goodwill generated which can be encashed in future, increasing market share with an eye on specially designed pricing policy which ultimately gives profits to the company, etc.

2.3 ENVIRONMENT IN, WHICH DECISIONS ARE MADE

Decision-maker faces the following situations while making decisions:

(*a*) **Decision under conditions of certainty**

This is a hypothetical situation in which complete information about the future business environment is available to the decision-maker. It is very easy for him to take a very good decision, as there is no uncertainty involved. But in real life, such situations are never available.

(*b*) **Decision under conditions of uncertainty**

The future state of events is not known, *i.e.,* there are more than one state of nature. As these uncertainties increase, the situation becomes more complex. The decision-maker does not have sufficient information and cannot assign probabilities to different occurrences.

(*c*) **Decision under risk**

Here, there are a number of states of nature like the above case. The only difference is the decision-maker has sufficient information and can allot probabilities to the different sates of nature *i.e.,* the risk can be quantified.

Decision Under Certainty

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This is a rare situation and no decision-maker is so fortunate to have complete information before making a decision. Hence, it is not a real life situation and is of no-consequence in managerial decisions.

In this situation only one state of nature exits and its probability is one. With one state of nature, possible alternatives could still be numerous and the decision-maker may use techniques like Linear programming, Transportation and Assignment technique, Economic Order Quantity (EOQ) model, input-output analysis, etc.

In our example of the manufacturing company, if the company had perfect information that the demand would be high: it would have three alternatives of expansion, construction of additional facilities and sub-contracting. Any one alternative, which gives the best payoff, say constructing additional facilities may be picked up to get the maximum benefit. So, the job of decision-maker is simple just to pick-up the best payoff in the column of state of nature (high demand, low demand, no demand) and use the associated alternative (expand, add facilities, sub-contract).

Decision Under Uncertainty

Under conditions of uncertainty, one may know the state of nature in future but what is the probability of occurrence is not known. Since the data or information is incomplete the decision model becomes complex and the decision is not optimal or the best decision. Such situations and decision problems are called the *Games Theory*, which will be taken up subsequently.

Let us take the case of our manufacturing company. If the company wishes to launch a new product like a DVD player, it knows that the demand of DVDs in future is likely to rise, but the probability that it will increase is not known. Also, the company may face the uncertainty of manufacturing these profitably, because the imported DVDs may become very cheap because of the Government policy.

There are number of criterion available for making decision under uncertainty. The assumption, of course, is that no probability distributions are available under these conditions. The following are discussed in this chapter :

(*a*) The maximax criterion

(*b*) The minimax (Maximin) criterion

(*c*) The Savage criterion (The Minimax Regret Criterion)

(*d*) The Laplace criterion (Criterion of Rationality)

(*e*) The Hurwicz criterion (Criterion of Realism).

In the above criteria, the assumption is also made that the decision-maker does not have an ‘intelligent’ opponent whose interest will oppose the interest of decision-maker. For example, when two armies fight each other, they are a case of intelligent opponents and such cases are dealt with and handled by Games Theory.

The Maximax Criterion

This the case of an optimistic criterion in which the decision-maker finds out the maximum possible payoff for every possible alternative and chooses the alternative with maximum payoff in the particular group. Let us reproduce the case of our manufacturing company with payoffs in rupee values.

Alternatives	State of nature (Demand)				Maximum of row
	High Demand	Moderate Demand	Low Demand	No Demand	
Expansion	40000	20000	–20000	–50000	40000
Add New Facilities	50000	25000	–30000	–70000	50000
Sub-contract	30000	20000	–5000	–20000	30000

Maximax ←

It is obvious that maximum payoff is Rs. 50000 corresponding to the alternative 'Add new facilities'.

The Minimax (Maximin) Criterion

The criterion is considered the most conservative as it aims at making the best of the worst possible situation. The decision-maker finds the minimum possible payoff and then worst possible payoff and then selects the best (maximum) out of minimum payoff. Let us review the above table with minimum of row and then pickup the maximum out of these, *i.e.,* –20000.

Alternative	State of nature (Demand)				Minimum of row
	High Demand	Moderate Demand	Low Demand	No Demand	
Expansion	40000	20000	–20000	–50000	–50000
Add New Facilities	50000	25000	–30000	–70000	–70000
Sub-contract	30000	20000	–5000	–20000	–20000

Maximin ←

The Savage (Minimax Regret) Criterion

This criterion is named after L.J. Savage who developed it. The decision-maker may regret making decision after it has been made because of the turn of events (state of nature). Hence, the decision-maker must keep the regret at the back of his mind and try and minimize that regret. It involves three steps process.

(*a*) Find out the quantum of regret associate with each alternative for different states of nature ;

(*b*) Determine the maximum regret for each alternative;

(*c*) Select the alternative which gives minimum of the maximum regrets determined in (*b*) above.

The criterion which is considered 'less conservative' as compared to Minimax criterion, which may described as extremely conservative. This quality of Savage criterion justifies the need of such a criterion. Let us consider the following loss matrix which may be quoted as a classic example of the illogical conclusion, whih minimax criterion can give.

Alternative	Nature state (demand)	
	θ_1	θ_2
a_1	Rs. 10000/-	Rs. 100/-
a_2	Rs. 8000/-	Rs. 8000/-

If we apply minimax criterion to this matrix, we are to pick-up a_2 alternative. However, commonsense dictates us to select a_1 as in this case maximum of Rs. 100/- may be lost where

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as in a_2 alternative there will be a certain loss of Rs. 8000/-. In Savage criterion this anomaly has been rectified by constructing a new loss matrix. New matrix is constructed by finding the difference between the best choice in the column and the particular value in the same column. In our above example in the column θ_1 the best choice is Rs. 8000. Now, the difference of the first value under θ_1 is 2000 and the second value is 0. Similarly, the best choice under θ_2, is Rs. 100. So, the difference of the first value under θ_2 is 0 and the second value is Rs. 7900/-. The minimax criterion yields a_1 as is expected. The students must note that the regret function represents loss and so the minimax and not the maximin criterion can be applied to this matrix:

Example 2.1. *Let us consider the following cost matrix which is to be conerted into he regret matrix:*

	θ_1	θ_2	θ_3	θ_2
a_1	5	10	18	25
a_2	8	7	8	23
a_3	21	18	12	21
a_4	30	22	19	15

Solution.

	θ_1	θ_2	θ_3	θ_4
a_1	0	3	10	10
a_2	3	0	0	8 Minimax value ←
a_3	16	11	4	6
a_4	5	15	11	0

The Laplace Criterion

Decision under uncertainty is represented in the form of a matrix, the columns of which represent the future state of nature and rows representing the alternatives or actions that are possible. Associated with each state of nature and each alternative action is the outcome or result of the action when a particular future state of nature occurs. This outcome evaluates the gain (or loss). Hence if a_1 represents the ith action ($i = 1, 2, \ldots, m$) and θ_1 represents the j^{th} nature state ($j = 1, 2, 3\ldots, n$), then V (a_1, θ_1) will represent the outcome resulting by ith action when θ_j state of nature occurs. This can be represented in the matrix below.

	θ_1	θ_2	....	θ_n
a_1	$v(a_1, \theta_1)$	$v(a_1, \theta_2)$		$v(a_1, 6_n)$
a_2	$v(a_2, \theta_1)$	$v(a_2, \theta_2)$		$v(a_2, \theta_n)$
⋮	⋮	⋮		M
⋮	⋮	⋮		M
a_m	$v(a_m, \theta_1)$	$v(a_m, \theta_2)$	...	$v(a_m, \theta_n)$

The probability of occurrence of $\theta_1, \theta_2, \ldots, \theta_n$ state of nature are not known, *i.e.,* are uncertain. If these probabilities were not different, we could determine them and the situation will not have sufficient reason to believe. $\theta_1, \theta_2, \ldots, \theta_n$ are equally likely to occur. This is called the **Principle of insufficient reason.** Under these circumstances one selects the alternative θ_i yielding the largest xpected gain. Hence,

$$\max a_i \left\{ \frac{1}{n} \sum_{j=1}^{n} v\left(a_i, \theta_j\right) \right\} .$$

where *i/n* is the probability that $\theta_1 (j = 1, 2 \ldots, n)$ occurs.

Example 2.2. *A service garage must decide on the level of spare parts it must stock to meet the need of arrival of cars for servicing. The exact number of cars arriving for serving is not known but it is expected to be in one of the four categories 80, 100, 120, 150 cars. Four levels of stocking are thus suggested with level being the best from the point of view of incurring cost if the number of cars arriving for servicing falls in category one. Any change from this ideal level will result in additional costs either because extra spares are stocked or because demand of servicing cannot be satisfied. The table below gives thse costs in thousand of rupees.*

Category of cars arriving for servicing

		θ_1	θ_2	θ_3	θ_4
	a_1	5	10	18	25
Stocking level	a_2	8	7	8	23
	a_3	21	18	12	21
	a_4	30	22	19	15

Solution. Laplace criterion assumes that probability of occurrence of $\theta_1, \theta_2, \theta_3$ and θ_4 are equal. Hence the probability of occurring of $p(\theta_1) = 1/4, j = 1, 2, 3, 4$. The expected costs associated with alternatives a_1, a_2, a_3, a_4 are as follows:

$$E[a_1] = \{1/4 (5 + 10 + 18 + 25)\} = 14.5$$
$$E[a_2] = \{1/4 (8 + 7 + 8 + 23)\} = 11.5$$
$$E[a_3] = \{1/4 (21 + 18 + 12 + 21)\} = 18.0$$
$$E(a_4)] = \{1/4 (30 + 22 + 19 + 15)\} = 21.5$$

Thus, the best level of stocking of spare parts in the service station according to Laplace criterion is given by a_2 (11.5).

Hurwicz Criterion

The criterion represents the method of choosing an alternative or action under the most optimistic or under the most pessimistic conditions. The decision-maker selects a parameter α (alpha), which is known as the **index of optimism.** When $\alpha = 1$, the criterion is too optimistic and similarly when $\alpha = 0$ it is too pessimistic. Depending upon the attitude of the decision-maker whether he leans towards optimism or pessimism, a value of α between 1 and 0 can be selected by him. When there is no strong inclination, one or the other, he may assume $\alpha = 1/2$.

Under most optimistic conditions

$$\max a_i \{ \alpha \max v(a_i \theta_j) + (1 - \alpha) \min v(a_i \theta_j)\}$$

If $v(a_i \theta_j)$ represents profits and

$$\min a_i \{\alpha \min v(a_i \theta_j) + (1 - \alpha) \max v(a_i \theta_j)\}$$ if $v(a_i \theta_j)$ represents costs.

Example 2.3. *Let us take the example of service garage given above and apply Hurwicz principle to it.The solution is provided below.*

	Min $v(a_i \theta_j)$	Max $v(a_i \theta_j)$	α Min $v(a_i \theta_j) + (1 - \alpha)$ Max $v(a_i \theta_j)$
a_1	5	25	15
a_2	7	23	15
a_3	12	21	16.5
a_4	15	30	22.5

In this example $\alpha = 1/2$ has been assumed. The optimum solution is provided by a_1 or a_2 (15).

Decision-making Under Conditions of Risk

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In real life situations managers have to make-decisions under conditions of risk. In decision-making under conditions of uncertainty, the decision-maker does not have sufficient information to assign probability to different states of nature. Whereas in decision-making under conditions of risk, the decision-maker has sufficient information to assign probabilities to each of the states of nature.

Decisions under risk are usually based on one of the following criterion :

(*a*) Expected value criterion (Expected monetary value –EMV criterion)

(*b*) Combined expected value and variance

(*c*) Known aspiration level

(*d*) Most likely occurrence of a future state

Each of the above criterion is discussed in the following paragraphs :

Expected Monetary Value criterion

This criterion consists of the following steps :

(*a*) Constructing a payoff matrix with different states of nature and alternative decisions. Enter the conditional profit for each decision-nature state combination along with the probabilities of occurrence of state and by adding the conditional values.

(*b*) Expected monetary value (EMV) is calculated for each decision by multiplying the profits by the probabilities of occurrence of the nature state and by adding the conditional values.

(*c*) Selecting the alternative, which yields highest EMV.

Example 2.4. *A newspaper boy has the following prbabilities of selling a magazine:*

No. of copies sold	***Probability***
10	0.10
11	0.15
12	0.20
13	0.25
14	0.30
	1.00

Cost of copy is 30 paise and sale price is 50 paise. He cannot return unsold copies. How many copies should he order ?

Solution Step I. *Constructing the conditional profit table*

It is obvious that the newspaper boy has to order between 10 and 14 copies. Since the sale price is 50 paise and the cost of one copy is 30 paise, he makes profit of 20 paise on each sale. If he sells 10 copies and he stocks 10 copies, he makes a profit of 10×20 paise = 200 paise. If he stocks 10 copies and demand is of say 12 copies, he still makes only a profit of 200 paise. When he stocks say 11 copies, his profit will be 220 paise but if only 10 copies are sold, his profit of 200 paise is reduced by 30 paise, the cost of unsold copy, *i.e.,* the profit is only 170 paise. Similarly, when he stocks 12 copies, the profit can be 240 paise, when he sells 12 copies, but if he sells only 11 copies, his profit must be reduced by one unsold copy, *i.e.,* $11 \times 20 - 30 \times 1 = 190$ paise. And if he stocks 12 copies but sells only 10, the profit of $20 \times 10 = 200$ paise must be reduced by $30 \times 2 = 60$ paise as these are two unsold copies, *i.e.,* $200 - 60 = 140$ paise. Hence payoff is equal to 20 paise × copis sold – 30 paise × unsold copies

No. of copies that can be sold	Probability	Possible stock action (copies)				
		10	11	12	13	14
10	0.10	200	170	140	110	80
11	0.15	200	220	190	160	130
12	0.20	200	220	240	210	180
13	0.25	200	220	240	260	230
14	0.30	200	220	240	260	280

Conditional Profit Table in Paise

Step II. Determine the expected value of each decision by multiplying the profit with the associated probability and by adding the values of all the alternatives. This is shown in the xpected Profit table drawn below.

No. of copies that can be sold	Probability	Expected profit (paise) by stocking				
		10	11	12	13	14
10	0.10	20	17	14	11	8
11	0.15	30	33	28.5	24	19.5
12	0.20	40	44	48	42	36
13	0.25	50	55	60	65	57.5
14	0.30	60	66	72	78	84
Total expected profit		200	215	222.5	220	205

Step III. Pick-up the alternative yielding highest EMV, which is 222.5 paise it means that the newspaper boy must order 12 copies to earn highest possible daily average profit of 222.5 paise.

Example 2.5. *You are given the following payoff of three acts A_1,A_2, A_3 and the events E_1, E_2, E_3:*

States of Nature	*Acts*		
	A_1	A_2	A_3
E_1	*25*	*–10*	*–125*
E_2	*400*	*440*	*400*
E_3	*650*	*740*	*750*

The probability of the state of nature are respectively 0.1, 0.7 and 0.2. Calculate and tabulate EMV and conclude which of the acts can be chosen as best.

Solution. EMV can be obtained by multiplying the probabilities with payoffs and adding all the values ofa particular action A_1, A_2 or A_3.

State of nature	Probability	Acts		
		A_1	A_2	A_3
E_1	0.1	$25 \times 0.1 = 2.5$	$-10 \times 0\ .1 = -1$	$-125 \times 0.1 = -12.5$
E_2	0.7	280	308	280
E_3	0.2	130	148	150
EMV		412.5	455	417.5

The best alternative is A_2.

Example 2.6. *A management is faced with the problem of choosing one of three products for manufacturing. The potential demand for each product may turn out to be good, moderate or poor. The probabilities for each of acts of nture were estimates as follows :*

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	Nature of Demand		
Product	***Good***	***Moderate***	***Poor***
X	*0.70*	*0.20*	*0.10*
Y	*0.50*	*0.30*	*0.20*
Z	*0.40*	*0.50*	*0.10*

The estimated profit or loss under he three states may be written as

Product	***Rs.***	***Rs.***	***Rs.***
X	*30000*	*20000*	*10000*
Y	*60000*	*30000*	*20000*
Z	*40000*	*10000*	*–15000*

Prepare the expected value table and advise the management about the choice of product.

Solution. Step I. Contruct the condition profit table.

Product	**Demand**			
	Probability Profit	**Good**	**Moderate**	**Poor**
X		0.70 30000	0.20 20000	0.10 10000
Y		0.30 6000	0.30 30000	0.20 2000
Z		0.40 4000	0.50 10000	0.10 –15000

Step II. Clculation of the expected values.

Product	**Demand**			**Expected Value**
	Good	**Moderate**	**Poor**	
X	0.70 × 30000	0.20 × 2000	0.10 × 1000	28000
Y	0.50 × 6000	0.30 × 3000	0.20 × 2000	43000 ←
Z	0.40 × 4000	0.50 × 1000	0.10 × (–15000)	19500

Step III. Select the best alternative. As the expected value of product Y is the highest, the management should choose this product.

Expected Opportunity Loss (EOL) Criterion

Another method is to determine minimum Expected Opportunity Loss (EOL). It represents the amount by which the maximum possible profit under various possible actions will be reduced. That course of action, which reduces this loss to minimum, is the best alternative. The following steps are involved in calculating EOL.

Step I. Prepare the conditional profit table for every action-state of nature combination and list the associated probabilities.

Step II. For each alternative find out the Conditional Opportunity Loss (COL) this is done by subtracting the payoff from the maximum payoff from a particular event.

Step III. COL's are multiplied by the respective probabilities. All these are added to give EOL.

Step IV. Select the alternative, which gives the minimum EOL.

Let $\theta_1, \theta_2, \theta_3$ be the states of nature and $p(\theta_1), p(\theta_2)..., p(\theta_n)$ be the respective probabilities of occurrence of these states of nature. Then the Expected Opportunity Loss (EOL) to acts a_1, a_2, a_3...will be

$$a_1 = (M_1 - p_{11})\, p(\theta_1) + (M_2 - p_{12})\, p(\theta_2) + ... + (M_n - p_{1n})\, p(\theta_n)$$
$$a_2 = (M_2 - p_{21})\, p(\theta_2) + (M_2 - p_{22})\, p(\theta_2) + ... + (M_n - p_{2n})\, p(\theta_n)$$

where M_1 = maximum profit or payoff corresponding to θ_i and $p_{11}, p_{12}, p_{13},..., p_{1n}$ be the outcomes of acts a_1, a_2 and so on.

Expected opportunity Loss (EOL) is also called Expected Value of Regrets (EVR).

Example 2.7. *Consider the following payoff table. The probability of occurrence of states of nature θ_1, θ_2, θ_3 and θ_4 are 0.25, 0.4, 0.15 and 0.20. Write the regret table an find out EOL of acts a_1, a_2 and a_3.*

Alternatives	**Nature State**			
	θ_1	θ_2	θ_3	θ_4
a_1	18	10	12	8
a_2	16	12	10	10
a_3	12	13	11	12

Solution. Step I. Conditional profit table has already been provided in the example:

Step II. Preparing COL table by subtracting the va1lue of each payoff in column θ_i (i = 1, 2, 3, 4) from the larest payoff value in the same column.

Acts	**State of Nature**				
		θ_1	θ_2	θ_3	θ_4
	Probability	0.25	0.4	0.15	0.20
a_1		18–18 = 0	13 – 10 = 3	12 – 12 = 0	12 – 8 = 4
a_2		18 – 16 = 2	13 – 12 = 1	12 – 10 = 2	12 – 10 = 2
a_3		18 – 12 = 6	13 – 13 = 0	12 – 11 = 1	12 – 12 = 0

Step III. EOL of acts a_1, a_2, a_3 are as follows:

$$a_1 = 0 \times .25 + 3 \times 0.4 + 0 \times .15 + 4 \times .20 = 2.25$$
$$a_2 = 2 \times .25 + 1 \times 0.4 + 2 \times 0.15 + 2 \times 0.20 = 1.6$$
$$a_3 = 6 \times .25 + 0 \times 0.4 + 1 \times .15 + 0 \times .20 = 1.65$$

Step IV. a_2 EOL is minimum = 1.6

Expected Value of Perfect Information (EVPI) Criterion

If the decision-maker had perfect information before taking the decision, this criterion provides the expected or average return in the long run.

Step I. Calculate the Expected-Payoff with Perfect Information (EPPI) which is equal to (Max. payoff in first state of nature × probability of occurrence of same state of nature) +

(Max payoff in second state of nature × probability of occurrence of same state of nature) + ... so on up to last state of nature.

Step II. Determine EVPI = EVPI – Maximum EMV.

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Example 2.8. *Payoff of three acts A, B and C ad state of nature XYZ are given below.*

Acts	*Payoff (In Rs.)*		
	A	*B*	*C*
State of Nature X	*–20*	*– 50*	*200*
Y	*200*	*–100*	*–50*
Z	*400*	*600*	*300*

The probability of the state of nature are 0.3, 0.4, and 0.3. Calculate the EMV for the data given and select the best act. Also find the expected value of perfect information (EVPI).

Solution. Step I. Calculating EMV of acts A, B, C

$$A = -20 \times 0.3 + 200 + 0.4 + 400 \times 0.3 = 194$$
$$B = -50 \times 0.3 - 100 \times 0.4 + 600 \times 0.3 = 125$$
$$C = 200 \times .3 - 50 \times .4 + 300 \times .3 = 130$$

Rs. 194 is he maximum EMV

Step II. Calculate EPPI

Nature of State	**Acts**				**Max for state of nature**	**Max Payoff × probability**
	Prob.	**A**	**B**	**C**		
X	0.3	–20	50	200	200	200 × 0.3 = 60
Y	0.4	200	– 100	– 50	200	200 × 0.4 = 80
Z	0.3	400	600	300	600	600 × 0.3 = 18
EPPI						320

Step III. EVPI – EMV

$$= 320 - 194 = \text{Rs. } 126$$

Example 2.9. *The probability of mnthly sales of an item is as follows :*

Monthly Sales (Units)	*0*	*1*	*2*	*3*	*4*	*5*	*6*
Probability	*0.01*	*0.06*	*0.25*	*0.30*	*0.22*	*0.10*	*0.06*

The cost of carrying inventory (unsold during the month) is Rs. 30 per cent per month and cost of unit storage is Rs. 70. Determine optimum stock to minimize expected cost.

Solution. Let P – Units purchased during a month

S – Units sold in a month

Then the cost function = Rs. 70 (S – P) if $P < S$

and = Rs. 30 (P – S) if $P \geq S$

Cost table for the aboveproblem can be constructed as follows:

Monthly Sales (Units) S	Probability	Strategies P (Units in stock or purchased)						
		0	1	2	3	4	5	6
0	0.01	0	30	60	90	120	150	180
1	0.06	70	0	30	60	90	120	150
2	0.25	140	70	0	30	60	90	120
3	0.30	210	140	70	0	30	60	90
4	0.22	280	210	140	70	0	30	60
5	0.10	350	280	210	140	70	0	30
6	0.06	420	350	280	210	140	70	0
Expected Cost		224	155	92	54	46	60	84

NOTES

The columns under different strategies are filled out using the cost functions given above. For example, under column P = 0, S = 0, the cost function is 0. When the monthly sales S = 1 and P = 0 since P < 0, cost function Rs. 70 (S – P) = 70 (1 – 0) = 70. Again under P = 3 and S = 1 since P > S cost function is Rs. 30

(P – S) = 30 (3 – 1) = Rs. 60 and so on.

$$\text{Expected cost} = \Sigma p_i \times C_i$$

where p_i is the probability of occurrence of sales and C_i is the cost incurred.

Expected cost = 0.01 × 0 + 0.06 × 70 + 0.25 × 140 + 0.30 × 210
(When 0 units are stocked) + 0.22 × 280 + 0.10 × 350 + 0.06 × 420 = Rs. 224

Expected cost = 0.01 × 30 + 0.06 × 0 + 0.25 × 70 + 0.30 × 140
(When 1 unit is stocked) + 0.22 × 210 + 0.10 × 280 + 0.06 × 350 = Rs. 155

Expected cost = 0.01 × 60 + 0.06 × 30 + 0.25 × 0 + 0.30 × 70
(Where 2 units are stocked) + 0.22 × 140 + 0.10 × 210 + 0.06 × 280 = Rs. 92

Expected cost = 0.01 × 90 + 0.60 + 0.25 × 30 + 0.30 × 0
(Where 3 units are stocked) + 0.22 × 70 + 0.10 × 140 + 0.06 × 210 = Rs. 54

Expected cost = 0.1 × 120 + 0.06 × 90 + 0.25 × 60 + 0.30 × 30
(Where 4 units are stocked) + 0.22 × 0 + 0.10 × 70 + 0.06 × 140 = Rs. 46

Expect cost = 0.1 × 150 + 0.06 × 20 + 0.25 × 90 + 0.30 × 60
(Where 5 units are stocked) + 0.22 × 30 + 0.10 × 0 + 0.06 × 70 = Rs. 53.75

The expected cost is minimum, *i.e.,* Rs. 46 if 4 units are stocked each month hence the optimum units to be stocked to minimize cost is 4.

Example 2.10. *The demandfor a seasonal product is given below.*

Demand during the season	*Probability*
40	*0.10*
45	*0.20*
50	*0.30*
55	*0.25*
60	*0.10*
65	*0.05*

The product costs Rs. 60 per unit and sells at Rs. 80 per unit. If the units are not sold within the season, they will have no market value.

(i) *Determine the optimum number of units to be produced.*

(ii) *Calculate* ***EVPI*** *and interpret it.*

Solution. (*i*) The payoff matrix can be prepared by remembering that

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Payoff = sale units × cost of product

For example, if 45 units are stocked and only 40 are sold, payoff = 40 × (0 – 35) × 60

= 3200 – 2700 = Rs. 500/-

Demand S	Probability	Strategies (P Units in stock or purchased)					
		40	**45**	**50**	**55**	**60**	**65**
0	0.10	800	500	200	–100	– 400	–700
45	0.20	800	900	600	300	0	–300
50	0.30	800	900	1000	700	400	100
55	0.25	800	900	1000	1100	800	500
60	0.10	800	900	1000	1100	1200	900
65	0.05	800	900	1000	1100	1200	1300
Expected value		800	860	840	54	460	180

It can be seen that expected value is the highest, *i.e.,* 860 with 45 units. Hence, the optimum number of units to be purchased is 45.

(*ii*) EPPI— The student should recall that this can be found out by multiplying probability with the maximum payoff under each demand so

$EPPI = 800 \times 0.1 + 900 \times 0.2 + 1000 \times 0.3 + 1100 \times .25 + 1200 \times 0.1 + 1300 \times 0.05$

$= Rs.\ 1020$

$EVPI = Rs.\ 1020 - 860 = Rs.\ 160$

Interpretation of EVPI

EVPI helps the decision-maker to get the perfect information about the state of nature. This helps in reducing the uncertainty. How much can be spent by the decision-maker to get perfect information ? EVPI gives that upper-limit. In the above example, the best alternative is to produce 45 units with expected payoff of Rs. 860. The expected profit under perfect information is Rs. 1020 and EVPI is 1020 – 860 = 160. However, the decision-maker can spend up to Rs. 160 per season to get the perfect information, which helps him to reduce the uncertainty of demand.

Example. 2.11. *Jagdamba Dairy wants to determine the quantity of ice cream it should produce to meet the demand. Past pattern of demand of their brand of ice cream is as follows :*

Quantity Demanded (kg)	*No. of Days Demand Occurred*
10	*8*
15	*12*
20	*20*
25	*60*
25	*40*
30	*40*
40	*40*
50	*20*

The company cannot stock ice cream more than 50 kg. Ice cream costs Rs. 60 and is sold at Rs. 70 per kg.

(a) *Construct a conditional profit table.*

(b) *Determine the alternative, which gives the maximum expected profit.*

(c) *Determine EVPI.*

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Solution. (*a*) Constructing conditional profit table.

It is clear from the problem that the company will not produce ice cream less than 10 kg and more than 50 kg.

Let CP denote the conditional profit, S the quantity in stock and D the demand, then CP = 10 S when D ≥ S because Rs. 10/- is the profit per kg.

CP = 70 D – 60 S when D ≤ S as whatever is the demand is sold at Rs. 70 per kg and what is not sold but is in stock and has been produced at the cost of Rs. 60 per kg must be reduced from the profit.

Let us take the case when the stock is 20 kg and the demand is 15 kg. In this case CP = 70 × 15 – 60 × 20 = – 150 and so on. Also, probability associated with demand levels have to be found out. The quantity of ice cream required for 8 days out of a total of 200 days is 10 kg. It means that the demand of 10 1kg has an associated probability of $\frac{8}{200}$ = 0.04. Similarly, other probabilities can also be determined. Conditional profit table along with associate probabilities is shown in table below.

Demand (kg)	**Probability**	**Possible alternatives of stock (kg)**						
		10	**15**	**20**	**25**	**30**	**40**	**50**
10	0.04	100	– 200	– 500	– 800	–1100	– 1700	– 2300
15	0.06	100	150	– 150	– 450	– 750	– 1350	– 1950
20	0.1	100	150	200	– 100	– 400	–1000	– 1600
25	0.3	100	150	200	250	– 50	– 650	– 1250
30	0.2	100	150	200	250	300	– 300	– 900
40	0.2	100	150	200	250	300	400	– 200
50	0.1	100	150	200	250	300	400	500

(b) Expected payof and EMV are shown in the table below :

Demand kg	**Probability**	**Possible alternatives of stock (kg)**						
		10	**15**	**20**	**25**	**30**	**40**	**50**
10	.04	4	–8	–20	–32	– 44	– 68	– 92
15	.06	6	9	–9	–27	– 45	– 81	–117
20	0.1	10	154	20	–10	– 40	–100	–160
25	0.3	30	45	60	75	– 15	–195	–375
30	0.2	20	30	40	50	– 60	– 60	–180
40	0.2	20	30	40	50	60	80	– 40
50	0.1	10	15	20	25	30	40	50
	EMV	100	136	151	–19	–114	–384	–914

Expected payoffs are determined by multiplying the payoff under each stock action by its associated probability. It means payoff for stock action of 10 is obtained by multiplying 100 by 0.04, *i.e.,* 4. Similarly, for stock 15, the probability 0.04 is multiplied with – 200, *i.e.,* 8 and so on.

Since the maximum EMV is Rs. 151 for stock of 20 kg of ice cream the dairy can expect an average daily profit of Rs. 151.

(c) *EVPI can be calclated with the help of following table:*

NOTES

Demand	Probability	Conditional Profit	Expected Profit with Perfect Information (EPPI)
10	0.04	100	4
15	0.06	150	9
20	0.1	200	20
25	0.3	250	75
30	0.2	300	60
40	0.2	400	80
50	0.1	500	50

EPPI = 298

EVPI = EPPI – EMV = 298 – 151 = Rs. 147/-

EMV for items having salvage value

In earlier calculations if the product is not sold it is assumed that it is of no use, *i.e.,* it has no salvage value. It may be true in many perishable products, but in case of other products this assumption is not correct as every such product will have a salvage value, hence, this salvage value must be taken into account while calculating the conditional profits for every stock options.

Example 2.12. *Let us continue with the above example and assume that unsold ice cream can be sold at Rs. 50 per kg. Post-sales pattern is between 10 – 13 kg per day. Find the EMV if the pas sales have the following probabilities:*

Sales	*10*	*11*	*12*	*13*
Probability	*0.2*	*0.2*	*.25*	*.35*

Solution. Let CP = Conditional profit

S = Quantity in stock

D = Market demand

CP = 10 S when D ≥ S when = 70 D – 60 S + 50 (S – D) when D < S

Payoff matrix usig the above relationship is as follows :

Demand or event	**Probability**	**Possible stock (S) action (alternative) in Rs.**			
		10	**11**	**12**	**13**
10	0.2	100	– 10	80	70
11	0.2	100	110	100	90
12	0.25	140	130	120	110
13	0.35	160	150	120	130

Now, we can calculate the expected pyoffs and the EMV for each stock action.

Possible Demand-D (event)	**Probability**	**Possible stock (S) action (alternative)**			
		10	**11**	**12**	**13**
10	0.20	0.2 × 100 = 20	0.2 × – 10 = –2	0.2 × 80 = 16	0.2 × 70 = 14
11	0.20	24	22	20	18
12	0.25	35	32.50	30	27.50
13	0.35	56	52.50	42	45.50
EMV (Rs)		135	105	108	105

Max EMV = Rs. 135 for stock action of 10 kg ice cream per day.

Example 2.13. *Daily demand (X) for bread at a general store is given bythe following probability distribution:*

NOTES

X	*100*	*150*	*200*	*250*	*300*
Probability	*0.20*	*0.25*	*0.30*	*0.15*	*0.10*

If a bread is not sold the same day it can be disposed of at Rs. 2 per piece at the end of the day. Otherwise, the price of fresh bread is Rs. 10. The cost of the bread is Rs. 8. If the optimum level of stocking is 200 breads daily, find out.

(a) Expected Monetary Value (EMV) of this optimum stock level

(b) Expected Value of Perfect Information (EVPI)

Solution. Selling = Rs. 10 (if sold the same day)

= Rs. 2 (if sold at the end of the day)

Let CP = Conditional profit

Then CP = 2S when D ≥ S

And CP = 10 D – 8S + 2 (S – D) when D < S

Conditional profit table ad EMV is calculated in the table below.

Demand (D)	**Probabilities**	**Payoff for stocking 200 breads (S)**	**Expected payoff**
100	0.20	– 400	– 80
150	0.20	0	0
200	0.30	400	120
200	0.30	400	120
250	0.15	800	120
300	0.10	1200	120
EMV			280

Let us calculate payoff value when demand (D) is 100 and stocking (S) is 200 since D < S

CP = 10 D – 8 S + 2 (S – D)

= 10 × 100 – 8 × 200 + 2 (200 – 100)

= 1000 – 1600 + 200

= – 400

When D = 150, S = 200

CP = 1500 – 1600 + 2 × 50

= 0

When D = 200, S = 200

CP = 400

When D = 250, S = 200 CP = 2500 – 1600 – 2 × 50 = 800

When D = 300, S = 200 CP = 1200

Expected payoff for 100 demand = 0.20 × 400 = – 80

150 demand = 0 and so on

Therefore, EMV = Rs. 280

If the demand is known with certainty, Expected Profit with Perfect Information EPPI) is calculated in the table below :

Demand (D)	Probabilities	Payoff for stocking 200 breads (S)	Expected payoff
100	0.20	200	40
150	0.25	300	75
200	0.30	400	120
250	0.15	500	75
300	0.10	600	60
EPPI			370

NOTES

$$\begin{aligned} \text{EVPI} &= \text{EPPI} - \text{EMV} \\ &= 370 - 280 \\ &= \text{Rs. } 90/- \end{aligned}$$

Decision Trees

We have seen decision situations in which no future decisions will depend on the decision taken now. Such decision criteria are called *'single-stage' alternatives*. In real life situations a decision taken has implications for the subsequent decisions. Hence one must consider multiple stage decision process in which the future decisions will depend on the decision taken now. Such decision problems can be represented graphically with the help of **decision tree,** such graphical representation facilitates the decision-making process.

Decision tree indicates decision alternatives, states of nature, probabilities associated with each state of nature and conditional profit or loss. It consists of nodes and branches. The following symbols are used:

Decision Node ▭ Square

State of nature ○ (circle)

Different courses of action or strategies emerge out of the nodes as main branches of the decision tree. At the end of each decision branch, there is a node representing state of nature out of which sub-branches come out representing change events. The payoffs from those alternatives and their probability of occurrence are shown alongside these branches. At the end of terminal of the chance branch is shown the expected value of the outcome.

Let us assume a decision-making problem representedby the following conditional profit table :

State of nature	Probability	Alternatives of production units	
		A_1 (50)	A_2 (100)
S_1 (High demand)	0.4	5000	10000
S_2 (Low demand)	0.6	6000	– 4000

The decision tree can be draw as follows to represent the above problem.

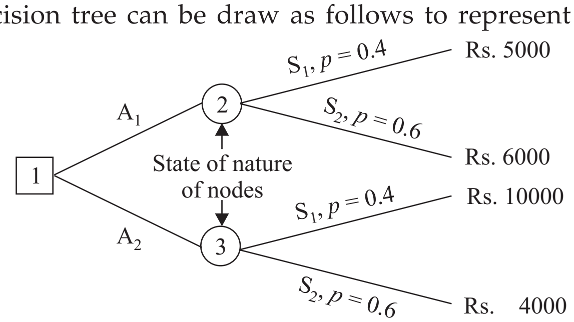

Fig. 2.1

EMV of alternative A_2 or node 3 is

$$= \text{Rs. } [10000 \times 0.4 + (-4000) \times 0.6]$$
$$= \text{Rs. } [4000 - 240]$$
$$= \text{Rs. } 1600/-$$

Some more illustrations are taken to demonstrate the use of decision tree.

Example 2.14. *A company has the option of building a new plant or expanding its existing plant. The decision depends primarily on the future demands for the product the plant will manufacture. The construction of a new plant can be justified on the grounds that if the demand keeps expanding the new plant can be run to its optimum capacity, otherwise it may be advisable to expand the existing facilities as the demand increases. The problem is shown wth the help of decision tree diagram below.*

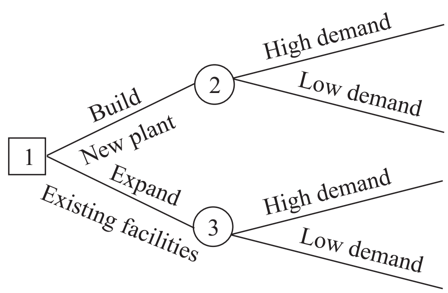

Fig. 2.2

Steps in Decision Tree Analysis

1. List the decision points and the strategies (alternative courses of action) for each decision point in a systematic manner.
2. Determine the probability and payoff associated with each alternative for each decision point.
3. Find out the expected payoff (EMV) of each course of action, starting from the extreme right working backward to the left.
4. Select the course of action that gives the best payoff for each alternative.
5. Continue working backward to the next decision point on the left.
6. Continue with this process until the first decision point on the extreme left is reached.
7. Consider the situation on the whole and find its course of action to be adopted from the beginning to the end under different possible outcomes.

Advantages of the Decision Tree Approach

1. It is systematic, orderly, logical and sequential approach.
2. It lists all possible outcomes and helps decision-nodes to examine each one of them.
3. It is easy to understand. Its graphical representation can be communicated to others with ease.
4. Since a decision now affects the decision-making in future, decision trees are particularly useful in such situations.
5. This approach can be applied to different decision problems, such as introduction of new product, investment decisions, etc.

Limitation of Decision Tree Approach

NOTES

1. In real life situations, the decisions are made under a large number of variables. In such cases, the diagram becomes extremely complicated.
2. It assumes utility of money is linear with time, which is not the case.
3. Decision Trees yields only and 'average' value solution as the problem is analyzed on the basis of expected values.
4. The assignment of probabilities for different events is many a times not exact and only a reasoned value.

Example 2.15. *A farmer is not sure whether he should dig a tube well in his field. He is presently using the canal water for irrigation of his fields for which he pays Rs. 5000 per year. The history of tube well digging in the village has not been very encouraging, only 50 per cent of the wells dug up to 200 feet yielded water. Some farmers drilled further up to 300 feet but only 25 per cent of them struck water at 300 feet. The cost of drilling is Rs. 100 per feet. The farmer has to make the following three decisions :*

(a) *Shall he drill up to 200 feet ?*

(b) *If no water is struck at 200 feet should he drill up to 300 feet ?*

(c) *Should he continue to buy water from government for next 5 years, as the life of the tube well is only five years ?*

Solution. The decision tree diagram of the above problem is drawn below :

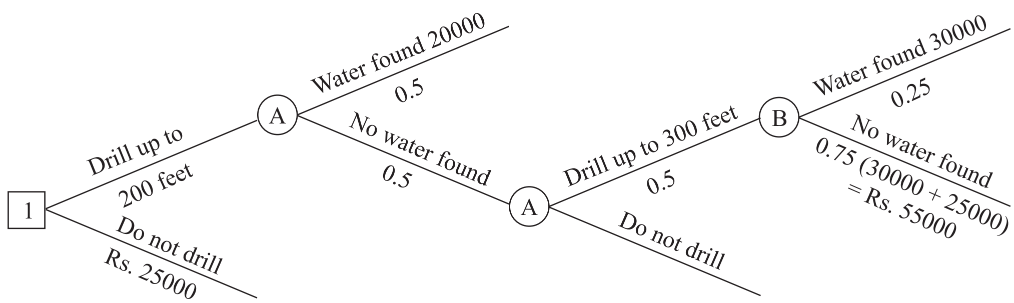

Fig. 2.3

At decision node 1, the farmer has to take a decision before drilling up to 200 ft., or not drilling. If he drills he pays Rs. 25000 @ Rs. 5000 per year for 5 years. If he drills up to 200 ft., there are two probabilities 0·5 of water found and no water being struck. If the water is found, the cost he incurs is Rs. 20000 as he digs 200 feet @ Rs. 100 per ft. If no water is found at 200 ft., he takes the decision of drilling up to 300 feet or not drilling. If he does not drill 300 ft., he incurs expanses of Rs. 45000 because he has already spent Rs. 20000 for drilling up to 200 feet and he has to pay Rs. 25000 @ Rs. 5000 per year. If he drills up to 300 ft., there is an assured probability of 0.25 that water will be found and of 0.75 that water will not be found. If water is found he spends Rs. 100 per ft. for 300 ft. If it is not found he spend Rs. 55000 as he has already spent Rs. 30000 on digging up to 300 ft., but he has also to spend Rs. 25000 @ Rs. 5000 per year for five years.

As explained earlier, in such problems we work backward

EMV of node B = $0.25 \times 30000 + 0.75 \times 55000$

= Rs. (7500 + 41250) = Rs. 48750

EMV of node 2 = Rs. 45000 (Choosing the lesser of the two of Rs. 48750 and Rs. 45000)

EMV of node A = Rs. [0.5 × 20000 + 0.5 × 48750] = Rs. (10000 + 24375)
= Rs. 34375

EMV of node 1 = 25000 (lesser of the two values Rs. 34375 and Rs. 25000)

NOTES

Hence, it can be easily seen the best course of action for the farmer is not to drill and pay Rs. 25000/ for water from canal to the government for five years.

Example 2.16. *A complex airborne navigating system incorporates a sub-assembly, which unrolls a map of the flight plan synchronously with the movement of the aeroplane. The sub-assembly is bought on very good terms from a sub-contractor, but is not always in perfect adjustment on delivery. The sub-assemblies can be readjusted on delivery to guarantee accuracy at a cost of Rs. 50 per sub-assembly. It is not however, possible to distinguish visually these sub-assemblies that need adjustments.*

Alternatively, the sub-assemblies can each be tested electronically at a cost of Rs. 10 per sub-assembly tested. Past experience shows that about 30 per cent of those supplied are defective; the probability of a test indicating a bad adjustment of the sub-assembly is 0.8, while the probability that the test indicates a good adjustment when the sub-assembly is properly adjusted is 0·7. If the adjustment is not made and the sub-assembly is found to be faulty when the system has its final check, the cost of subsequent rectifications will be Rs. 140.

Draw up an appropriate decision tree to show the alternatives open to the purchaser and use it to determie his appropriate course of action.

Solution.

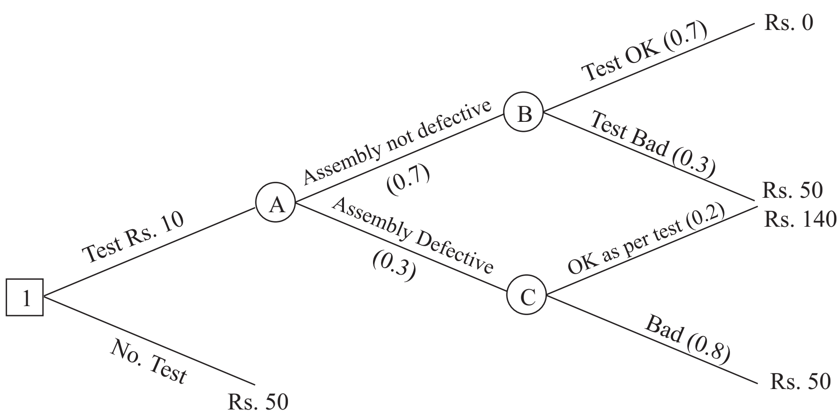

Fig. 2.4

The purchaser can either have the sub-assembly tested electrically and pay Rs. 10 or can have it readjusted at a cost of Rs. 50. If he gets them tested, probability of defective sub-assembly is 0.3 and good is 0.7. Those which are defective out of these 80% (0.8) will be with bad adjustment and need readjustment at the cost of Rs. 50. Only 20% (0.2) are OK as per test but if these are found to be faulty in the final check, the purchaser has to spend Rs. 140. Out of 70% of sub-assemblies, which are not defective, 70% are OK as per test and such assemblies will not cost anything to the purchaser but 30%, which are bad as per test, will have to be readjusted at the cost of Rs. 50 per sub-assembly.

EMV of node B = Rs. (0.7 + 0.350) = Rs. 15

EMV of node C = Rs. (0.2 × 140 + 0.8 × 50) = Rs. 68

EMV of node A = Rs. (0.7 × 15 + 0.3 × 68) = Rs. 30.90.

EMV of node 1 = Rs. (10 + 30.90) = 40.90. (When test is carried out)

EMV of node 1 = Rs. 50 (when no test is carried out)

Since the least cost is Rs. 40.90; the decision the purchaser has to take is to get sub-assemblies tested.

NOTES

Example 2.17. *XYZ Ltd., wants to update/change its existing manufacturing prices for product A. It wants to strengthen its R & D cell and conduct research for finding a better product of manufacturing which can get them higher profits. At present the company is earning a profit of Rs. 20000 after paying for material, labour and overheads. XYZ Ltd., has the following four alternatives :*

(*a*) *The company continues with the existing process.*

(*b*) *The company conducts research P, which costs Rs. 20000, has 75% probability of success and can get the profit of Rs. 5000.*

(*c*) *The company conducts research Q, which costs Rs. 10000, has 50% probability of success and can get the profit of Rs. 25000.*

(*d*) *The company pays Rs. 10000 as royalty for a new product and can get profit of Rs. 20000.*

The company can carryout only one out of the two types of research P and Q because of certain limitations. Draw a decision tree diagram and find the best strategy for XYZ Ltd.

Solution. The decision tree is drawn as shown below :

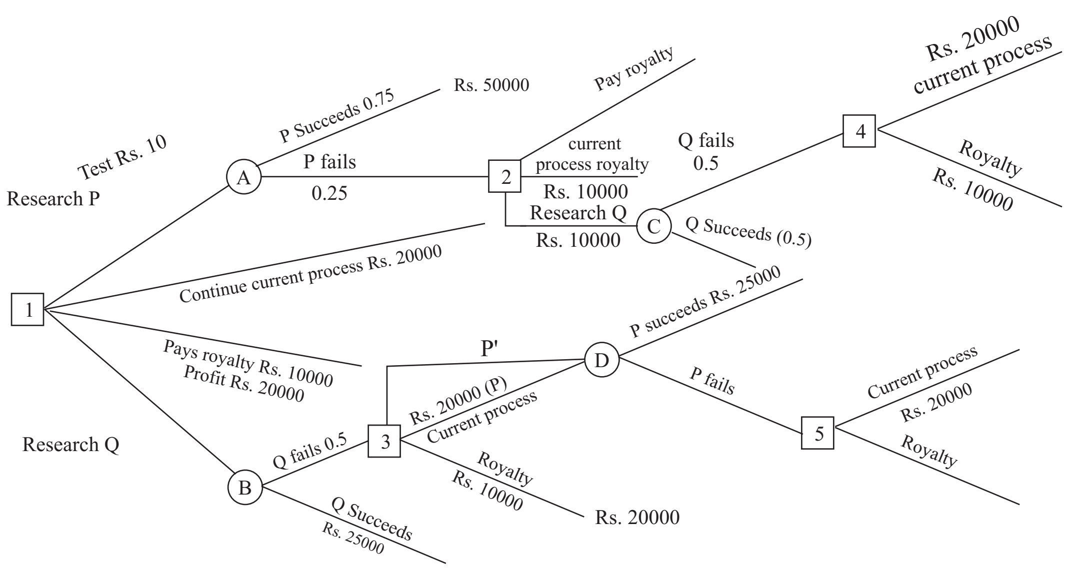

Fig. 2.5

Points 1, 2, 3, 4 and 5 are decision boxes. The four alternatives to the company are shown coming out of decision box 1. Research P succeeds (probability 0.75) or it fails (Probability 0.25). If it fails the company has three alternatives; conduct research Q, continue with the existing process or pay royalty. If Q fails the company is left only with the option of paying royalty or continuing with existing process. The payoffs are written at the end of the branches.

Let us now calculate the EMV starting from node 5 (working backward)

EMV at decision node 5 = Maximum out of

(*a*) Rs. 20000

(*b*) Rs. 20000 – 10000 = 10000 = Rs. 20000

EMV at decision node 4 = Maximum out of B Rs. 20000 and Rs. (20000 – 10000)

= Rs. 20000

EMV at decision node C = Rs. (0.5 × 20000 + 0.5 × 20000) = Rs. 22500

EMV at decision node D – Rs. (0.75 × 50000 + 0.25 × 20000) = Rs. 42500

EMV at decision node 3 = Maximum out of

(*a*) Rs. 20000

(*b*) Rs. 20000 – 10000 = Rs. 10000 ?

EMV at decision node 2 = As for decision node 3, *i.e.,* Rs. 20000

EMV at decision node B = Rs. (0.5 × 20000 + 0.5 × 20000) = Rs. (22500)

EMV at decision node A = Rs. (0.75 × 50000 + 0.25 × 20000) = Rs. 42500

EMV at decision node 1 = Maximum out of four alternatives

(*a*) Rs. (42500 – 20000) = Rs. 22500

(*b*) Rs. (22500 – 10000) = 12500

(*c*) Rs. 20000

(*d*) Rs. (20000 – 10000) = Rs. 10000

Max. = Rs. 22500

The company should conduct research P to find a new process to earn a maximum profit of Rs. 22500.

Example 2.18. *ABC Lt., has invented a picture cell phone. It is faced with selecting one alternative out of the following strategies :*

(a) Manufacture the cell phone.

(b) Take royalty from another manufacturer.

(c) Sell the rights for the invention and take a lump sum amount.

Profit in thousands of rupees which can be incurred and the probability associated with sch alternatives are shown in the table below :

Represent the company problem in the form of the decision tree and suggest what decision the comany should take to maximize profits.

Solution.

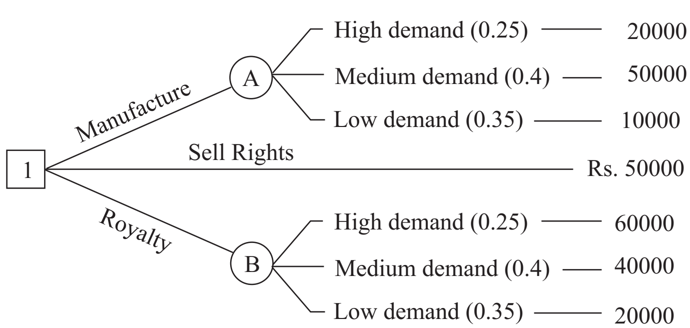

Fig. 2.6

EMV at node B = Rs. (0.25 × 60000 + 0.4 × 40000 + .35 × 20000)
= Rs. (15000 + 16000 + 7000) = Rs. 38000

EMV at node A = Rs. (0.25 × 200000 + 0.4 × 50000 + 0.35 × – 10000)
= Rs. (50000 + 20000 – 3500) = Rs. 66500

EMV at decision node 1 = Maximum out of Rs. 50000, 38000, 66000
= Rs. 66500

NOTES

Thus, the best decision by ABC Ltd., is to manufacture the picture cell phone itself to get profit of Rs. 66500.

Example 2.19. *The investment staff of TNC Bank is considering four investment proposals for clients, shares, bonds, real estate and saving certificates; these investments will be held for one year. The past data regarding the four proposals is given below :*

Shares : *There is 25% chance that shares will decline by 10%, 30% chance that they will remain stable and 45% chance that they will increase in value by 15%. Also the shares under consideration do not pay any dividends.*

Bonds : *These bonds stand a 40% chance of increase in value by 5% and 60% chance of remaining stable and they yield 12%.*

Real Estate : *This proposal has a 20% chance of increasing 30% in value, a 25% chance of increasing 20% in value a 40% chance of increasing 10% in value, a 10% chance of remaining stable and a 5% chance of loosing 5% of its value.*

Saving Certificates : *These certificates will yields 8.5 with certainty.*

Use a decision tree to structure the alternatives available to the investment staff, and using the expected value criteria choose the alternative with the highest expected value.

Solution. Let us assume that we have Rs. 100 to invest. The decision tree is shown below.

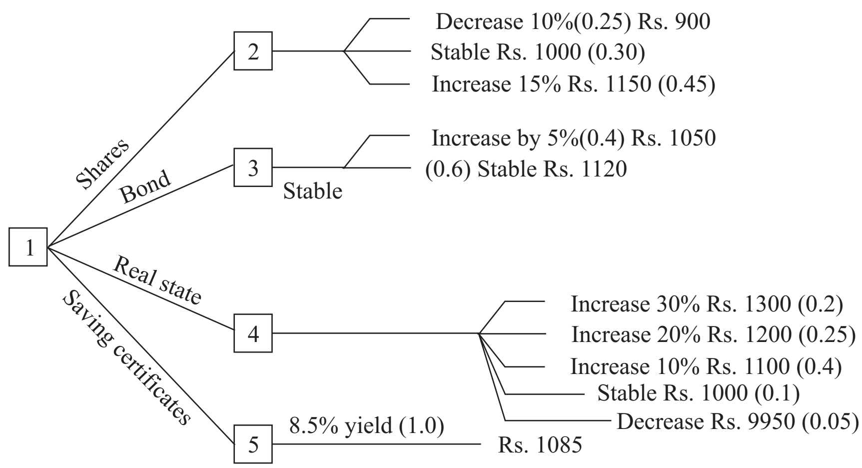

Fig. 2.7

EMV at node 5 = Rs. 1085

EMV at node 4 = Rs. $(0.2 \times 1300 + 0.25 \times 1200 + 0.04 \times 1100 + 0.05 \times 9950)$

= Rs. (260 + 300 + 440 + 497.50) = Rs. 1497.50

EMV at node 3 = Rs. $(0.4 \times 1370 + 0.06 \times 1120)$ = Rs. (548 + 672) = Rs. 1220

EMV at node 2 = Rs. $(0.25 \times 900 + 0.30 \times 1000 + 0.045 \times 1150)$

= Rs. (225 + 300 + 517.50) = Rs. 1042.50

EMV at decision node 4 is maximum, *i.e.*, 1497.50. So the decision should be to invest in real estate.

2.4 SUMMARY

- Decision theory, provides a rational approach in dealing with such situations, where the information is incomplete and uncertain about future conditions.

- In decision theory, a number of statistical techniques can help the management in making rational decisions.
- It is very difficult to identify the exact events that may occur in future. However, it is possible to list all that can happen. The future events are not under the control of the decision-maker. In decision theory, identifying the future events is called the *state of nature*
- Under conditions of uncertainty, one may know the state of nature in future but what is the probability of occurrence is not known. Since the data or information is incomplete the decision model becomes complex and the decision is not optimal or the best decision.
- Decision under uncertainty is represented in the form of a matrix, the columns of which represent the future state of nature and rows representing the alternatives or actions that are possible
- In real life situations managers have to make-decisions under conditions of risk. In decision-making under conditions of uncertainty, the decision-maker does not have sufficient information to assign probability to different states of nature.
- EVPI helps the decision-maker to get the perfect information about the state of nature. This helps in reducing the uncertainty.
- In real life situations a decision taken has implications for the subsequent decisions. Hence one must consider multiple stage decision process in which the future decisions will depend on the decision taken now.
- Decision tree indicates decision alternatives, states of nature, probabilities associated with each state of nature and conditional profit or loss. It consists of nodes and branches

2.5 REVIEW AND DISCUSSION QUESTIONS

1. Explain clearly the various ingredients of a decision problem. What are the basic steps of a decision-making process ?

2. What are different environment in which decision are made ?

3. Explain clearly the following terms :

(i) Action space, (ii) State-of-nature, (iii) Payoff table, and (iv) Opportunity loss.

4. Describe some methods, which are useful for decision-making under uncertainty. Illustrate each by an example.

5. Write a short note on decision-making under uncertainty.

6. Indicate the difference between decision under risk and decision under uncertainty in statistical decision theory.

7. Write notes on (a) Laplace criterion, (b) Minimax regret criterion, (c) Criterion of Bayes, and (d) Criterion of realism.

8. With suitable illustrative examples, explain the maximum, and the regret criteria in decision-making.

9. (*a*) Write a note on Hurwicz criterion. Define a the index of optimism connected with it. For what values of a the criterion is too optimistic and too pessimistic.

(*b*) Define 'maximum criterion'. Illustrate with an example the 'Savage minimax regret criterion'.

NOTES

10. What is EMV ? How is it computed to be used as a criterion of decision-making and which ?

11. Define EVPI. How is it calculated ?

12. Explain the difference between EOL and EVPI.

13. Bring out the significance of “utility” as a superior decision criterion as compared to expected value criterion.

14. (*a*) What do you understand by Decision tree analysis ?

(*b*) What is node in a decision tree ?

15. A decision problem hasbeen expressed in the following payoff table :

	Outcome		
Action	**I**	**II**	**III**
A	10	20	26
B	30	30	60
C	40	30	20

(*a*) What is the minimum payoff ?

(*b*) What is the minimum opportunity loss function ?

16. A farmer wants to plan which of the three crops he should plant on his 100-acre form. The profit of each crop depends upon the rainfall during the growing season. The rainfall could be high, medium and low. The estimated profit of the former fr *each of the crops is as shown in the table :*

Estimated conditional Profit

Rainfall	**Crop A**	**Crop B**	**Crop C**
High	6000	3000	7000
Medium	4000	4500	4000
Low	2000	5000	5000

The farmer decides to plant only one crop, which would be his best crop use the following criterion :

(a) Maximum

(b) Maximin

(c) Laplace

(d) Minmax Regret.

UNIT 3: LINEAR PROGRAMMING-I

(Formulation of LPP and Graphical Solution Method)

NOTES

Structure

3.1 INTRODUCTION

Linear Programming (LP) is a mathematical technique, which is used for allocating limited resources to a number of demands in an optimal manner. When a set of alternatives is available and one wants to select the best, this technique is very helpful. Management wants to make the best use of organizational resources. Human resources, which may be skilled, semi-skilled or unskilled must be put to optimal use. Similarly, the material resources like machines must be used in an effective manner. Time is very important resource and any job must be completed in allotted time. Application of LP requires that the following conditions must be met:

(*a*) There must be a well-defined objective of the organization such as:

  (*i*) Maximizing profit

  (*ii*) Minimizing cost.

(b) This objective function must be expressed as a linear function of variables involved in decision-making.

(*c*) There must be a constraint on availability of resources for the objective functions, *i.e.*, for achieving maximum profit or for reducing the cost to a minimum.

LP technique establishes a linear relationship between two or more variables involved in management decisions described above. Linear means it is directly proportional, *i.e.*, if 5 per cent increase in manpower results in 5 per cent increase in output, it is a linear relationship.

(*d*) Alternative course of action must be available to select the best, for example, if a company is producing four different types of products and wants to cut down one product, which one should stop manufacturing. The problem gives rise to a number of alternatives and so LP can be used.

(*e*) Objective function must be expressed mathematically, *i.e.*, we must be able to develop a linear mathematical relationship between the objective and its limitation. Linear equations are of first degree, *i.e.*, if we want x and y as the variable, the equation $5x + 10y = 20$ is a linear equation in which x and y can assume different values. However, an equation like $5x^2 + 10y^2 = 200$ is not a linear equation, because of the variable x and y are squared, this is a typical second degree equation.

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Formulation of Linear Programming Problem

The formulation of linear programming requires the following steps:

(*a*) Identifying/defining the decision variables.

(*b*) Specifying/defining the objective function to be maximized or minimized.

(*c*) Identifying the constraint equations, which have to be expressed as equalities or inequalities.

(*d*) Using the equation either in graphical or simplex method to find out the value of decision variables to optimize the objective function.

Assumptions for Solving a Linear Programming Problem

The application of LP makes use of the following assumptions :

(*a*) *Linearity.* The objective function and each constraint is linear.

(*b*) *Certain and Constant.* It means that the number of resources available and production requirements are known exactly and remain constant.

(*c*) *Non-negative Variables.* The values of decision variables are non-negative and represent real life solutions. Negative values of physical goods or products are impossible. Production of minus 10 refrigerators is meaningless.

Potential Applications of Linear Programming

Some real life situations, where LP is very useful are given below:

1. **Product Mix Problem :** Organizations often face the problems of making decision to manufacture different quantities of products, with the constraint of manpower, machines, availability of raw materials, etc. The idea is to minimize cost of production or maximization of profit under a given set of conditions.
2. **Transportation Problems :** LP finds typical use in finding solution to such problems. The problem is to transport products from a number of sources to a number of destinations with minimum cost. These are the real life situations where the goods have to be moved from the factory premises to warehouses in different parts of the country or from warehouses to Clearing and Forwarding (C and F) agents and so on. How many goods should be transported to meet the demands of different destinations so that the cost of transportation is minimum, can best be decided by use of LP.
3. **Blending Problems :** Large number of products use different types and quantities of raw materials. For example, in textiles industry a number of raw materials are used. The idea is to make available different raw materials (with different specifications) in such quantities so that the product is manufactured at minimum raw material cost. Such problems are called *blending problems*.
4. **The Diet Problem :** This problem arises when one has to decide mixing of different type of foods to get a particular amount of nutritional values with minimizing cost of purchasing the diet. Hospitals can use LP methods for solving such problems.
5. **Investment Decisions (Portfolio selection) Problem :** This is a very common problem with those who want to make use of different investment opportunities. When the amount to be invested is fixed and different opportunities like investment in shares, bonds, mutual funds, post-office schemes, banks, etc., are available, LP method can provide us the solution to get maximum returns.
6. **Use of LP by Airlines :** Operation of airlines routes is a very complex problem. With

limited aircrafts and large number of destinations airlines would like to operate in the most economic routes at particular flight timings. LP is a very useful technique for solving such problems.

NOTES

3.2 FORMULATION OF LISUBHEAD 1NEAR PROGRAMMING PROBLEMS

Example 3.1. *A manufacturing company is producing two products A and B. Each of the products A and B requires the use of two machines P and Q. A requires 4 hours of processing on machine P and 3 hours of processing on machine Q. Product B requires 3 hours of processing on machine P and 6 hours of processing on machine Q. The unit profits for products A and B are Rs. 20 and Rs. 30 respectively. The available time in a given quarter on machine P is 1000 hours and on machine Q is 1200 hours. The market survey has predicted that 250 units of products A and 300 units of product B can be consumed in a quarter. The company is interested in deciding the product mix to maximize the profits. Formulate this problem as LP model.*

Solution. Formulating the problem in mathematical equations

Let X_A = the quantity of product of type A manufactured in a quarter.

X_B = the quantity of products of type B manufactured in a quarter.

Z = the profit earned in a quarter.

(Objective function, which is to be maximized).

Therefore, $Z = 20\ X_A + 30\ X_B$

Z is to be maximized under the following conditions :

$4X_A + 3X_B \leq 1000$ (Time constraint of machine P)

$3X_A + 6X_B \leq 1200$ (Time constraint of machine Q)

$X_A \leq 250$ (Selling constraint of product A)

$X_B \leq 300$ (Selling constraint of product B)

X_A and $X_B \geq 0$ (Condition of non-negativity).

Example 3.2. *An oil refinery uses blending process to produce gasoline in a typical manufacturing process. Crude A and B are mixed to produce gasoline G_1 and gasoline G_2. The inputs and outputs of the process are as follows :*

Process	***Inputs (tons)***		***Outputs (tons)***	
	Crude A	*Crude B*	*Gasoline (G_1)*	*Gasoline (G_2)*
1	1	2	6	8
2	6	8	5	7

Availability of crude A is only 200 tons and B 300 tons. Market demand of gasoline G_1 is 150 tons and gasoline G_2 is 120 tons. Profit by using process 1 is Rs. 200 per ton and by using process 2 is Rs. 250 per ton. What is the optimal mix of two blending processes so that the refinery can maximize its profits?

Solution. Let X be the number of tons to be produced by process 1.

Y be the number of tons to be produced by process 2.

Z = Profit earned (Objective function which is to be maximized) .

Therefore, $Z = 200X + 250Y$

Z is to be maximized under the following conditions :

$$X + 6Y \leq 200$$
$$2X + 8Y \leq 300$$
$$6X + 5Y \leq 150$$
$$8X + 7Y \leq 120$$
$$X \geq 0$$
$$Y \geq 0$$

NOTES

Example 2.3. *Manufacturing company XYZ Ltd. manufactures two different types of products, refrigerators and washing machines. Both these products have to be processed through two machines, Machine A and Machine B. Machine A is available for 200 hours and machine B is available for 100 hours. The requirement of time on these mahines is as follows :*

	Refrigerate	*Washing machine*
Machine A	*10*	*6*
Machine B	*5*	*4*

The company makes a profit of Rs. 800 on sale of one refrigerator and Rs. 500 on sale of one washing machine. What quantities of refrigerators and washing machines should company XYZ Ltd. produce to maximize its profits ?

Solution. Let X be the number of refrigerators to be manufactured and Y be the number of washing machines to be manufactured.

Z = Profit earned (Objective function which is to be maximized).

Therefore, $Z = 800X + 500Y$

Z is to be maximized subject to the following constraints :

$10X + 6Y \leq 200$ (Availability of machine A)

$5X + 4Y \leq 100$ (Availability of machine B)

$$X \geq 0$$
$$Y \geq 0$$

Example 3.4. *A company manufactures three types of electrical products, electric iron, fan and toaster. All the three products have to be processed on two machines A and B. The processing time required by each product on both the machines is as given below :*

	Electric Iron	*Fan*	*Toaster*
Machine A	2	3	2
Machine B	1	2	3

Machine A is available only for 200 hours and machine B is available for 160 hours. The firm should not manufacture more than 400 electric irons, more than 500 fans and more than 200 toasters. An electric iron gives a profit of Rs. 110, a fan of Rs. 150 and a toaster of Rs. 80. What product mix would you recommend to the company so that its profits are maximized ?

Solution. Let X_1 be the number of electric irons to be manufactured.

X_2 be the number of fans to be manufactured.

X_3 be the number of toasters to be manufactured.

Z = profits generated (Objective functions which is to be maximized).

Z is to be maximized under the following constraints :

$$Z = 110X_1 + 150X_2 + 80X_3$$

$$2X_1 + 3X_2 + 2X_3 \le 200$$
$$X_1 + 2X_2 + 3X_3 \le 160$$
Also, $$X_1, X_2, X_3 \ge 0$$

NOTES

Example 3.5. *A firm is manufacturing three products A, B and C. Time to manufacture product A is twice that for B and thrice that for C and they are to be produced in the ratio 2 : 3 : 4. The relevant data is given below. If the whole labour is engaged in manufacturing product A, 2000 units of the product can be produced. There is a demand for at least 200, 300 and 400 units of the product A, B, C and the profit earned per unit is Rs. 100, Rs. 70 and Rs. 50 respectively. Formulate the problem as a linea programming problem.*

Raw material	***Requirement per unit of product (kg)***			***Total availability (Kg)***
	A	*B*	*C*	
P	6	5	9	4000
Q	4	8	6	5000

Solution. Maximize Z = 100A + 70B + 50C (Objective function) subject to the following constraints:

$$6A + 5B + 9C \le 4000 \text{ (Constraint of raw material P)}$$
$$4A + 8B + 6C \le 5000 \text{ (Constraint of raw material Q)}$$
$$A + C \le 2000$$
$$A + \frac{1}{2}B + \frac{1}{3} \ge 200$$
$$A \ge 200$$
$$B \ge 300$$
$$C \ge 400$$

A: B: C : : 2 : 3 : 4 $$\frac{A}{2} = \frac{B}{2}, \quad .e., \quad 3A - 2B = 0$$
$$\frac{B}{3} = \frac{C}{4} \text{ or } 4B - 3C = 0$$
$$A, B, C \ge 0$$

Example 3.6. *A town located at high altitude has two locations where kerosene and petrol is stored by Army for use in four different zones during winters when the highway is closed and no supplies of kerosene and petrol are possible to these locations. The table below provides the cost (Rs.) of supplying one kilolitres of kerosene and petrol from each stock location to each zone. In addition, the storing location capacity and normal level of demand for each zone are indicated in kilolitres. Formulate the LP problem.*

	Zone				***Maximum Supply (k. Litre)***
	1	**2**	**3**	*4*	
Storage location 1	4	6	2.50	3.00	1000
Storage location 2	5	2	3.50	4.50	800
Storage location 3	300	500	400	350	

Solution. In this problem, there are eight decisions to be made-how many k. litres should be transported from each storage location to each zone. In some cases the best decision may be not to transport any units from a particular location to a particular zone.

Let x_{11} , x_{12} , x_{13} , x_{14}, ... denote the number of k. litres supplied by location 1 to zone 1, 1 to 2, 1 to 3 and 1 to 4 respectively.

Similarly, let x_{21} , x_{22} , x_{23} , x_{24} be the number of k. litres supplied by location 2 to zone 1, 2 to zone 2, 2 to zone 3 and 2 to zone 4.

NOTES

Total cost = $4 x_{11} + 6 x_{12} + 2.50 x_{13} + 3.50 x_{14} + 5 x_{21} + 2 x_{22} + 300 x_{23} + 4.50 x_{24}$.

This function has to be minimised.

The constraints are:

$$x_{11} + x_{12} + x_{13} + x_{14} \leq 1000 \text{ (For location 1)}$$
$$x_{21} + x_{22} + x_{23} + x_{24} \leq 800 \text{ (For location 2)}$$

Also, the constraint of ensuring that each zone receives the quantity demanded.

For zone 1, the sum of the transportation from location 1 and 2 should be 300 k. litres

or $$x_{11} + x_{21} = 300.$$

Given that each origin can supply units to each destination in some measure

$$x_{11} + x_{22} = 500$$
$$x_{13} + x_{23} = 400$$
$$x_{14} + x_{24} = 350$$

The complete formulation of LP model is as follows:

Minimize $$Z = 4 x_{11} + 6 x_{12} + 2.5 x_{13} + 3 x_{14} + 5 x_{21} + 2 x_{22} + 3.5 x_{23} + 4.5 x_{24}$$

Subject to
$$x_{11} + x_{12} + x_{13} + x_{14} \leq 1000$$
$$x_{21} + x_{22} + x_{23} + x_{24} \leq 800$$
$$x_{11} + x_{21} = 300$$
$$x_{12} + x_{22} = 500$$
$$x_{13} + x_{23} = 400$$
$$x_{14} + x_{24} = 350$$
$$x_{11}, x_{12}, x_{13}, x_{14}, x_{21}, x_{22}, x_{23}, x_{24} \geq 0.$$

3.3 GRAPHICAL METHOD OF SOLVING LINEAR PROGRAMMING PROBLEMS

The various steps in solving the LP problem using the graphical method are as follows :

1. Formulate the problem with mathematical form by:
   (*a*) Specifying the decision variables,
   (*b*) Identifying the objective function and
   (*c*) Writing the constraint equations.
2. Plot the constraint equations on a graph
3. Identify the area of feasible solution
4. Locate the corner points of the feasible region
5. Plot the objective function
6. Choose the points where objective functions have optimal value.

Example 3.7. *A manufacturing company is producing two products A and B. Each requires processing on two machines 1 and 2. Product A requires 3 hours of processing on machine 1 and 2 hours on machine 2. Product B requires 2 hours of processing on machine 1 and 6 hours on machine 2. The unit profits for product A and B are Rs. 10 and Rs. 20 respectively. The available time in a given quarter on machine 1 and machine 2 are 1200 hours and 1500 hours respectively. The market survey has predicted that not more than 400 units of product A and not more than 250 units of product B can be sold in the given quarter. The company wants to determine the product mix to maximize the profits. Formulate the problem as linear programming mathematical model and solve it graphically.*

Solution.

Step 1. *Formulating the problem into a mathematical model.*

Let X be the number of product A manufactured in a quarter and Y be the number of product B manufactured in a quarter.

$Z =$ profit in a quarter (objective function)

Z is to be maximized under the following constraints:

$Z = 10X + 20Y$

$3X + 2Y \leq 1200$ (Time constraint of machine 1)

$2X + 6Y \leq 1500$ (Time constraint of machine 2)

$X \leq 400$ (Selling constraint of product A)

$Y \leq 250$ (Selling constraint of product B)

$X \geq 0$

(Non-negativity constraint)

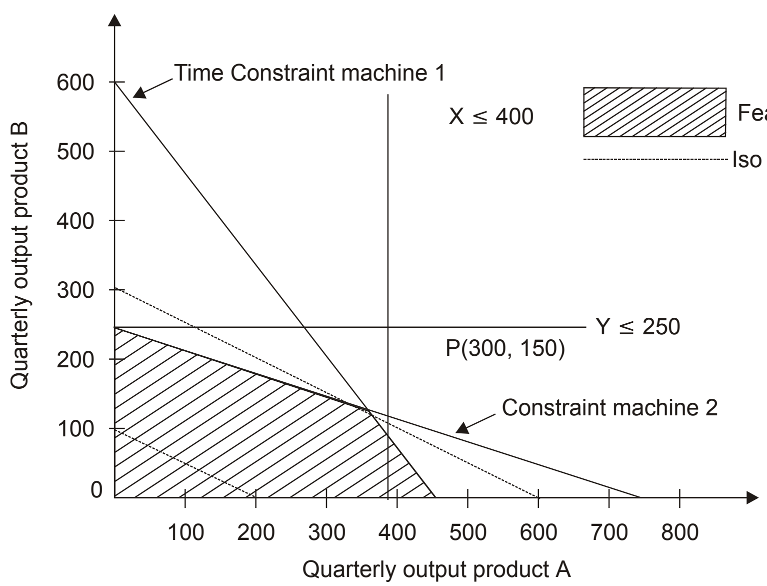


Fig. 3.1

Step 2. *Plot the constraint equations in the graph* as shown in Figure 3.1.

This can be done by converting the inequalities into equalities.

$3X + 2Y = 1200$

When $X = 0$

$Y = 600$

When $Y = 0$

$X = 400$

Hence, plot X = 400 and Y = 600 and draw a line joining these two points. Similarly, draw time constraint of machine 2. Also, the selling constraints of product A and product B can be drawn.

NOTES

Step 3. *Identify the area of feasible solutions* as shown by the shaded area in Figure 3.1.

Step 4. *Plot the objective function*. This can be done by assuming some arbitrary profit figures, which can be related within the feasible area (shaded area). For example, if we assume a profit of Rs. 2000, it can be obtained by manufacturing 200 units of product A or 100 units of product B. Plot 200 on product A axis and 100 on product B axis. When we join these two points we get the isoprofit line. It is called *iso* (same) profit line because any point on this line will give different combinations of product A and B, which will give the same profit, *i.e.*, Rs. 2000/-. Many different lines with different profits Rs. 4000, Rs. 6000, etc., can be plotted.

Step 5. *Determine the optimal solution.*

To do this, draw a straight line parallel to the isoprofit line already drawn, in such a manner that it is farthest from the origin but also intersects the feasible area at some point. This point where the line drawn parallel to the isoprofit line and which is farthest from the origin intersects the feasible area is called *optimal point* (P). In the present problem this point P is represented by P (300, 150).

It means 300 units of product A and 150 units of product B should be manufactured to give maximum profit to the company.

Maximum profit $Z = 10X + 20Y$

$X = 300$

$Y = 150$

$Z = 300 \times 10 + 20 \times 150 = \text{Rs. } 6000/-$

Example 3.8. *Solve the following inequalities graphically:*

Maximize $Z = 3X + 2Y$

Subject to $X + Y \leq 35$

$X - Y \geq 0$

$X \leq 20$

$Y \geq 5$

$X \geq 0$

$Y \geq 0$

Solution. Convert the above constraints inequalities into equalities.

$X + Y = 35$ ...(*i*)

$Y = 5$ ...(*ii*)

$Y = 15$ ...(*iii*)

$X - Y = 0$ ...(*iv*)

$X = 20$ ...(*v*)

From (*i*) When $X = 0, Y = 35$ (0, 35)

When $X = 35, Y = 0$ (35, 0)

From (*iv*) $X = 0$

Y = 0

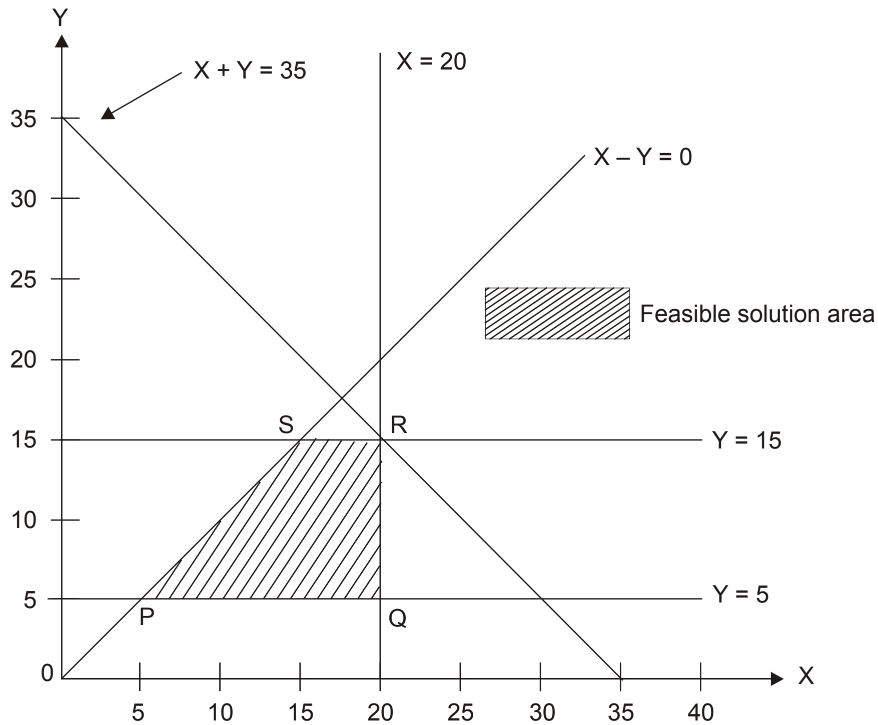


Fig. 3.3

The feasible solution region is shown with shaded area. Let us find out which point gives the maximum value of objective function Z.

Point	Coordinate	Max value 3X + 2Y
P	(5, 5)	15
Q	(20, 5)	70
R	(20, 15)	90
S	(15, 15)	75

Hence maximum value Z = Rs. 90 at point R.

Example 3.9. *A landlord has set up a stud farm where he wants to breed the best horses. His veterinary doctor has advised him to use two special diets, say A and B, for the horses. The nutrition value of these diets and minimum requirement of these nutrients is as follows:*

Nutrients	*Availability of nutrients in the products*		*Minimum requirement*
	A	***B***	
1	40	12	120
2	20	10	60
3	8	36	108

If special diet A costs Rs. 60 per unit and diet B costs Rs. 80 per unit, using LP graphics method, determine how much of products A and B must be purchased by the landlord so as to provide the horses not less than the minimum required as per the advise of the vet.

Solution. Mathematical formulation of the problem

Let X_1 and X_2 be the number of units of diet A and diet B to be purchased. The other data given in the problem can be summarized in the table below.

NOTES

Decision	Product	Nutrient available			Cost *(Rs)*
1	A	40	20	8	60
2	B	12	10	36	80
Minimum requirement		120	60	108	

Objective function

Minimize $Z = 60X_1 + 80X_2$ with the following constraints:

$$40X_1 + 12X_2 \geq 120$$
$$20X_1 + 10X_2 \geq 60$$
$$8X_1 + 36X_2 \geq 108$$
$$X_1, X_2 > 0$$

Plotting the constraint equations in the graph

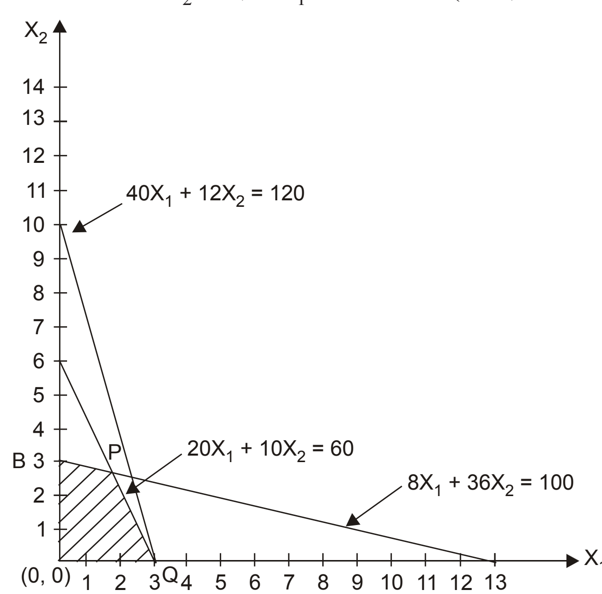
First, the inequalities have to be converted into equalities.

$$40X_1 + 12X_2 = 120 \quad ...(i)$$
$$20X_1 + 10X_2 = 60 \quad ...(ii)$$
$$8X_1 + 36X_2 = 108 \quad ...(iii)$$

From (*i*) $X_1 = 0,\ X_2 = 10$
$X_2 = 0,\ X_1 = 3$ (3, 10)

From (*ii*) $X_1 = 0,\ X_2 = 6$
$X_2 = 0,\ X_1 = 3$ (3, 6)

From (*iii*) $X_1 = 0,\ X_2 = 3$
$X_2 = 0,\ X_1 = 13.5$ (13.5, 3



, 3)

Fig. 3.3

These can be plotted in Figure 3.14 .

Coordinates of P can be found out by solving the equations

$$20X_1 + 10X_2 = 60 \text{ and } 8X_1 + 30X_2 = 108$$
$$X_1 = 1.7$$
$$X_2 = 2.6$$

Point	Coordinates	Objective function $60X_1 + 80X_2$
B	(0, 3)	240
P	(1.7, 2.6)	102 + 208 = 310
Q	(3, 0)	180

Since the minimum cost is at point Q, the landlord should purchase 1.7 units of product X1 and 2.6 units of product X_2 .

Example 3.10. *A firm manufactures two products P_1 and P_2 on which the profits earned are Rs. 5 and Rs. 8 respectively. Each product is prepared on two machines M_1 and M_2 , the machine time required for these products on the two machines and their availability is as shown elow.*

	Product P_1	*Product P_2*	*Availability of machine (minutes) per day*
Machine M_1	*2*	*1*	*400*
Machine M_2	*4*	*1*	*600*

Find the number of units of products P_1 and P_2 to be manufactured per day to get maximum profits.

Solution. Maximize $Z = 5P_1 + 8P_2$ (Objective function)

Subject to the following constraints:

$$2P_1 + P_2 \leq 400 \quad \text{...}(i)$$
$$4P_1 + P_2 \leq 600 \quad \text{...}(ii)$$
$$P_1 \geq 0$$
$$P_2 \geq 0$$

Convert the inequalities into equalities.

$$2P_1 + P_2 = 400 \text{...}(i)$$
$$4P_1 + P_2 = 600 \text{...}(ii)$$

From (*i*) $P_1 = 0$ $P_2 = 400$ (0, 400)

$P_2 = 0$ $P_1 = 200$ (200, 0)

From (*ii*) $P_1 = 0$ $P_2 = 600$ (0, 600)

$P_2 = 0$ $P_1 = 150$ (150, 0)

Plotting the equation on the

Shaded area is the feasible solution area. Coordinates of OABC and valueof Z.

Point	Coordinates	Value of $Z = 5P_1 + 8P_2$
O	(0, 0)	0
A	(0, 400)	3200
B	(100, 200)	2100
C	(150, 0)	750

Maximum value of Z = 3200 at point A.

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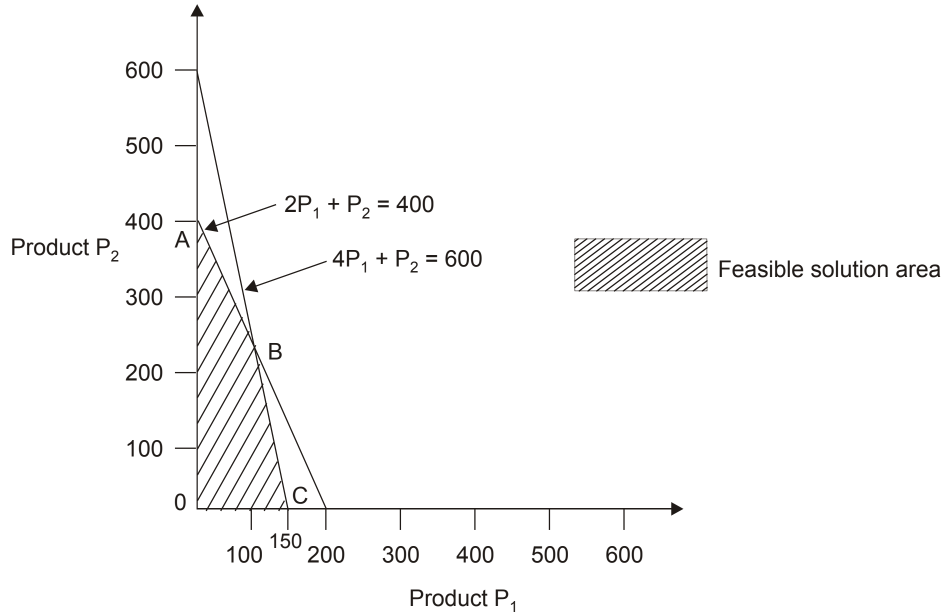

Fig. 3.4

Example 3.11. *The ABC company wishes to plan its advertising strategy. There are two medias under consideration call them Magazine I and Magazine II respectively. Magazine I has a reach of 2500 potential customers. The cost per page of advertising is Rs. 400 and Rs. 600 in magazine I and II respectively. The firm has a monthly budget of Rs. 6000. There is an important requirement that the total reach for income group under Rs. 20000 per annum should not exceed 4000 potential customers. The reach in magazine I and II for this income group is 400 and 200 potential customers. How many pages should be brought in the two magazines to maximize the total reach ?*

Solution. Let X and Y be the number of pages in magazines I and II respectively which ABC company should buy

Maximize Z = 2000X + 2500Y (Objective function as in this case the reach of two magazines has to be maximized)

Subject to the following constraints:

$400X + 600Y \leq 6000$ (Constraint of monthly budget)

$400X + 200Y \leq 4000$ (Constraint of minimum potential customer which should be reached)

$X > 0$

$Y > 0$

Now, convert the inequalities into equalities.

$400X + 600Y = 6000$ ...(*i*)

$400X + 200Y = 4000$ ...(*ii*)

From (*i*)	X = 0	Y = 10	(0, 10)
	Y = 0	X = 15	(15, 0)
From (*ii*)	X = 0	Y = 20	(0, 20)
	Y = 0	X = 10	(10, 0)

Plotting these equations as straight lines on the graph to find the feasible solution area.

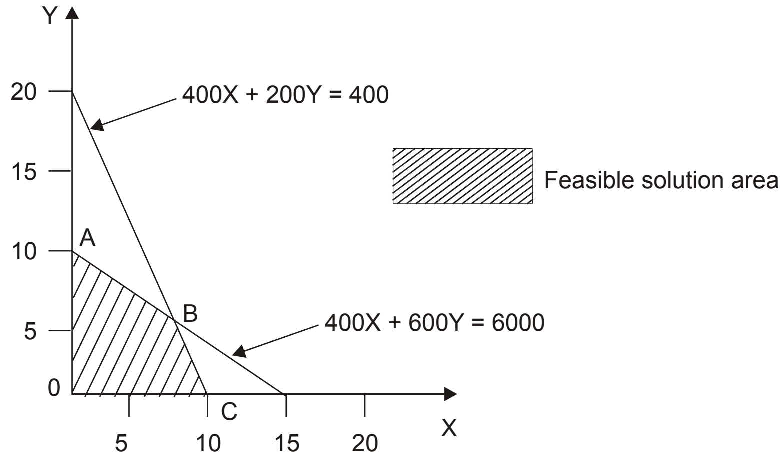

Fig. 3.5

OABC is the feasible solution area.

Coordinates of these points and value of Z is as follows:

Point	Coordinates	Value of Z = 2000X + 2500Y
O	(0, 0)	0
A	(0, 10)	25000
B	(7.5, 5)	27500
C	(10, 0)	25000

Coordinates of point B can also be found out by solving equations (*i*) and (*ii*) subtract (*ii*) from (*i*).

$$400Y = 2000$$

$$Y = 5$$

Putting Y = 5 in (*ii*) gives X = 7.5.

The company should produce 7.5 pages in magazine I and 5 pages in magazine II to maximize its reach, *i.e.*, to 27500 people.

Exceptional Cases. We have so far discussed the linear programming problems where there is a unique optimal solution always available. However, it may not be the case for all the problems. In general, a LPP should have the following:

(*a*) A definite and unique optimal solution.

(*b*) An unbounded solution.

(*c*) No solution.

Let us discuss the case of an unbounded solution.

Example 3.12. *A firm manufacture two products. The products must be processed through one department. Product A requires 4 hours per unit and product B requires 2 hours per unit. Total production time available for the coming week is 60 hours. A restriction in planning the production schedule, therefore, is that total hours used in producing two products cannot exceed 60 hours, or if* x_1 *equals the number of units produced of product A and* x_2 *equals the number of units produced of product B, the restriction is represented by the inequality.*

$$4x_1 + 2x_2 \leq 60$$

NOTES

There are two other restrictions implied by the variable definitions. Since each variable represents a production quantity, neither variable can be negative. These restrictions are represented by the inequalities $x_1 \geq 0, x_2 \geq 0$. The solution set of original inequality represents different combinations of the two products which can be manufactured while not exceeding the 60 hours limit. Figure 3.30 illustrates this graphically.

All combinations of the two products represented by points A and B would use all 60 hours.

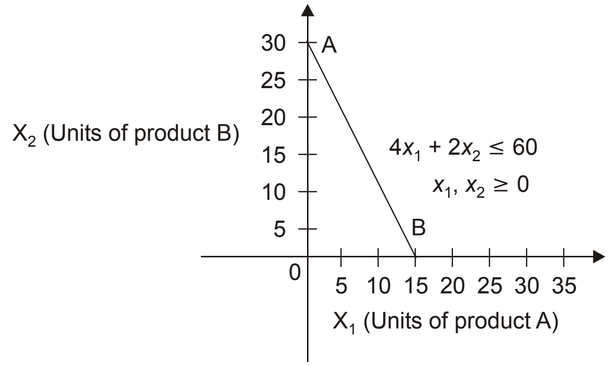

Fig. 3.6

Example 3.13. *Determine the optimal solution to the LPP given below graphically.*

Minimize $Z = 3x_1 + 6x_2$

Subject to $4x_1 + x_2 \geq 20$ ...(*i*)

$x_1 + x_2 \leq 20$ ...(*ii*)

$x_1 + x_2 \geq 10$ ...(*iii*)

$x_1, x_2 \geq 0$

Solution.

For equation (*i*) $x_1 = 0, x_2 = 20, x_2 = 0, x_1 = 5$

For equation (*ii*) $x_1 = 0, x_2 = 20, x_2 = 0, x_1 = 20$

For equation (iii) $x_1 = 0, x_2 = 10, x_2 = 0, x_1$

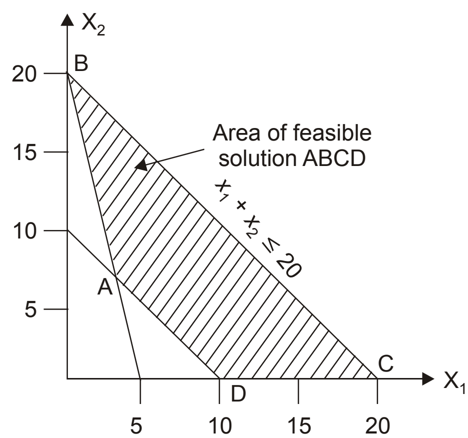

Fig. 3.7

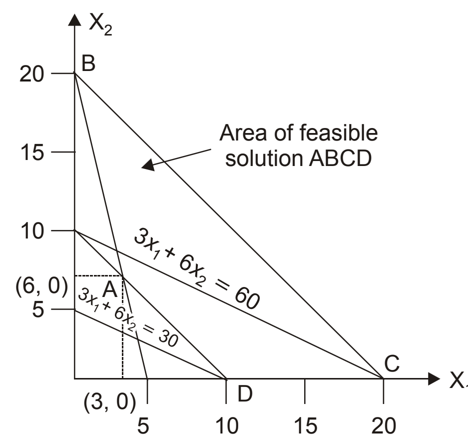

Fig. 3.8

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To find out the optimal solution, let us assume an arbitrary value for Z say 60.

$$Z = 3x_1 + 6x_2 = 60$$

$$x_1 = 0, x_2 = 10 \text{ and } x_2 = 0, x_1 = 20.$$

This is graphed on the right hand side of Figure 3.35.

To determine the direction of the movement of the objective function, we can choose a point on either side of the line $3x_1 + 6x_2 = 60$ and determine the corresponding value of Z. If we select the origin, the value of objective function at (0, 0) is

$$Z = 0.$$

Since the value of Z at origin is less than 60, our conclusion is that movement of the objective function towards origin results in lower value of Z. Since we want to minimize Z, we will want to move the objective function, parallel to itself, as close to the origin as possible while still having it touch a point in the area of feasible solution. The last point touched before the objective function moves out of the area of feasible solution is D (10, 0). Given that minimum value of Z occurs at (10, 0) the minimum value is

$$Z = 3 \times 10 + 6 \times 0 = 30.$$

SUMMARY

- Linear Programming (LP) is a mathematical technique, which is used for allocating limited resources to a number of demands in an optimal manner. When a set of alternatives is available and one wants to select the best, this technique is very helpful. Management wants to make the best use of organizational resources
- The formulation of linear programming requires the following steps:
  - (*a*) Identifying/defining the decision variables.
  - (*b*) Specifying/defining the objective function to be maximized or minimized.
  - (*c*) Identifying the constraint equations, which have to be expressed as equalities or inequalities.
  - (*d*) Using the equation either in graphical or simplex method to find out the value of decision variables to optimize the objective function.
- The various steps in solving the LP problem using the graphical method are as follows :
  1. Formulate the problem with mathematical form by:
     - (*a*) Specifying the decision variables,
     - (*b*) Identifying the objective function and
     - (*c*) Writing the constraint equations.
  2. Plot the constraint equations on a graph
  3. Identify the area of feasible solution
  4. Locate the corner points of the feasible region
  5. Plot the objective function
  6. Choose the points where objective functions have optimal value.

REVIEW AND DISCUSSION QUESTIONS

1. What are the essential characteristics of a Linear Programming model ?
2. Explain the terms: Key decision, objective, alternative and restrictions in the context of linear optimisation models by assuming a suitable industrial situation.
3. Discuss the application of LP in any functional area of management. Use suitable example from business or industry.
4. "Linear Programming is the most widely used and most successfully used mathematical approach to decision-making". Comment.

NOTES

5. Explain the advantages and major limitations of LP model. Illustrate your answer with suitable examples.

6. What are the major allocation problems that can be solved by using LP model ? Briefly discuss each one of them.

7. Illustrate graphically the following special cases of LP problems:

(*i*) Multiple optimal solutions

(*ii*) Non-feasible solution

(*iii*) Unbounded problem.

8. Write the standard form of LPP in matrix form.

9. Write a short note on the limitations of Linear Programming.

10. Discuss briefly the steps to formulate a Linear Programming Problem. Give suitable examples.

11. Explain the following terms:

(*i*) Linearity (*ii*) Feasible solution

(*iii*) Objective function (*iv*) Unbounded solution

(*v*) Optimal solution.

12. Maximize $P = 1.4\,X_1 + X_2$

Subject to $X_1 \leq 3$

$2X_1 + X_2 \leq 8$

$3X_1 + 4X_2 \leq 24$

$X_1 \geq 0, X_2 \geq 0$

Using graphic method.

13. Three products are produced on three different operations. The limit of available time for the three operations are respectively 430, 460 and 420 minutes and profit per unit for the three products are Rs. 3, 2 and 5 respectively. Times in minutes per unit on each machine operation is as follows :

Operation	**Product**		
	I	**II**	**III**
1	1	2	1
2	3	0	2
3	1	4	0

Write LP model for this problem.

14. Solve the LPP given below by graphical method and shade the region representing the feasible solution.

Minimize $Z = 2X_1 + 10\,X_2$

$X_1 - X_2 \geq 0$

$X_1 - 5X_2 \geq -5$

$X_1 \geq 0, X_2 \geq 0$

15. Solve graphically the following LPP :

Minimize $Z = 3X_1 + 5X_2$

Subject to $-3X_1 + 4X_2 \leq 12$

$2X_1 - X_2 \geq -2$

$2X_1 + 3X \geq 12$

$X_1 \leq 4$

$X_2 \geq 2$

$X_1 \geq 0, X_2 \geq 0$

16. Solve the LPP graphically:

Maximize $Z = 80X_1 + 120\, X_2$

Subject to $X_1 + X_2 \leq 9$

$20X_1 + 50X_2 \leq 360$

$X_1 \geq 2, X_2 \geq 3$

when X_1 and $X_2 \geq 0$

17. A small manufacturer employs 5 skilled men and 10 semi-skilled men and makes an article in two qualities, a deluxe model and an ordinary model. Making of a deluxe model requires 2 hours work by a skilled man and 2 hours work by a semi-skilled man. The ordinary model requires one hour work by a skilled man and 3 hours work by a semi-skilled man. By union rules no man can work for more than 8 hours a day. The manufacturer's clear profit of the deluxe model is Rs. 10 and of the ordinary model Rs. 8. Formulate the model of the problem.

18. A firm manufactures three products A, B and C. The profits are Rs. 3, Rs. 2 and Rs. 4 respectively. The firm has two machines and the processing time in minutes for each machine on each product is given below.

Machine	**Products**		
	A	B	C
C	4	3	5
D	2	2	4

Machines C and D have 2,000 and 2,500 machine-minutes respectively. The firm must manufacture 100 A's, 200 B's and 50 C's but not more than 150 A's set-up an LP model to maximize the profit.

19. A certain farming organisation operates three farms of comparable productivity. The output of each farm is limited both by useable acreage and by the amount of water available for irrigation. Following is the data for upcoming season:

Farm	**Useable Acreage**	**Water available in acre feet**
1	400	1500
2	600	2000
3	300	900

The organisation is considering three crops for planting which differ primarily in their expected profit per acre and their consumption of water. Furthermore, the total average that can be devoted to each of the crops is limited by the amount 8 appropriate harvesting equipment available.

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Crop	Minimum acreage	Water consumption in acre feet per acre	Expected point per acre
A	700	5	₹400
B	800	4	₹300
C	300	3	₹100

In order to maintain a uniform workload among the farms, it is the policy of the organisation that the percentage of useable acreage planted must be the same at each farm. However, any combination of the crops may be grown at any of the farms. The organisation wishes to know, how much of each crop should be planted at the respective farms in order to maximize expected profits. Formulate thus as a linear programming problem.

ILLUSTRATED CASE STUDIES

Case Study No 1

ABC Ltd engages Quality Control inspectors from a consultancy company which has a pool of such manpower as it does not want to have inspector on its own pay roll. It uses 40 inspectors of Grade I level and 60 inspectors of Grade II level. The company expects the following standards:

Grade of Inspectors	No of pieces to be inspected in an 8 hours day	Wages per hour (Rs.)	Accuracy to be achieved
I	30	25	98%
II	20	40	96%

At least 1600 pieces must be inspected in a day of 8 hours.

If an error in inspection is made, it will cost the company Rs 50. The company is interested in knowing the optimal assignment of inspectors so that the inspection costs are minimized.

Hint. Two types of costs are incurred by each grade of inspector

(*a*) Wages to be paid to the inspector.

(*b*) Cost of inspection error.

Cost of one grade I inspector per hour

$$\text{Rs. } \left(25 + 50 \times 0.02 \times \frac{30}{8}\right)$$

Similarly, cost of one grade II inspector/hour can be determined.

Minimize Z = x_1 × Cost of grade I inspector + x_2 × cost of grade II inspector

With the constraints $x_1 \leq 40$

$x_2 \leq 60$ (Constraint of number of inspectors)

$30x_1 + 20\,x_2 \leq 200$

Case Study No. 2.

A company produces special alloys used by the defence forces in manufacture of certain components of an air defence gun. The alloy specifications provided by the defence forces are:

(*a*) Chromium ≥ 10%

(*b*) Melting point ≥ 600°C

(*c*) Specific gravity ≤ 0.99

The company uses three types of raw materials to produce the alloy of above specifications. The properties of the raw material required for this purpose are as given below.

Parameter of Specification	**Property**		
	Raw material P	Raw material Q	Raw material R
Chromium (%)	8	12	16
Melting point (˚C)	700 ˚C	650 ˚C	600 ˚C
Specific Gravity	0.98	1.02	0.96

The cost of the raw materials per unit in open market are:

$$P = \text{Rs. } 10000$$
$$Q = \text{Rs. } 25000$$
$$R = \text{Rs. } 8000$$

The company wants stock to the specifications so that they can continue getting the orders from defence department, which is their mainstay for profits and also give them the credibility and goodwill in the market. At the same time, they are interested in reducing the cost of raw materials to the minimum. Advise the company as to what percentage of raw materials P, Q and R are to be used for making the alloy.

[**Hint.** Minimize Z = $10000\, x_1 + 25000\, x_2 + 8000\, x_3$

Constraints are :

$$8x_1 + 12x_2 + 16x \geq 10$$
$$700x_1 + 650\, x_2 + 600\, x_3 \geq 600$$
$$0.981\, x_1 + 1{\cdot}02\, x_2 + 0.96\, x_3 \leq 0{\cdot}99$$
$$x_1 + x_2 + x_3 = 100]$$

UNIT 4: LINEAR PROGRAMMING-II

(Simplex Method)

NOTES

Structure

4.1 INTRODUCTION

Simplex method is an algebraic procedure in which a series of repetitive operations are used and we progressively approach the optimal solution. Thus, this procedure has a number of steps to find the solution to any problem, consisting of any number of variables and constraints, however problems with more than 4 variables cannot be solved manually and require the use of computer for solving them.

This method developed by the American mathematician G. B. Dantizg, can be used to solve any problem, which has a solution. The process of reaching the optimal solution through different stages is also called iterative, because the same computational steps are repeated a number of times before the optimum solution is reached.

4.2 SOLVING OPERATIONS RESEARCH PROBLEM USING SIMPLEX METHOD

Following are various steps for solving OR problem using simplex method.

Step I. *Formulate the problem.*

The problem must be put in the form of a mathematical model. The standard form of the LP model has the following properties:

(*a*) An objective function, which has to be maximized or minimized.

(*b*) All the constraints can be put in the form of equations.

(*c*) All the variables are non-negative.

Step II. *Set-up the initial simplex table with slack variable or surplus variables in the solution.*

A constraint of type ≤ or ≥ can be converted into an equation by adding a slack variable or subtracting a surplus variable on the left hand side of the constraint.

For example, in the constraint $X_1 + 3X_2 \leq 15$ we add a slack $S_1 \geq 0$ to the left side to obtain an equation.

$$X_1 + 3X_2 + S_1 = 15, \quad S_1 \geq 0$$

Now, consider the constraint $2X_1 + 3X_2 - X_3 \geq 4$, since the left side is not smaller than the right side, we can subtract a surplus variable $S_2 > 0$ from the left side to obtain the equation.

$$2X_1 + 3X_2 - X_3 - S_2 = 4 \quad S_2 > 0$$

The use of the slack variable or surplus variable will become clear in the actual example as we proceed.

Step III. *Determine the decision variables which are to be brought in the solution.*

Step IV. *Determine which variables to replace.*

Step V. *Calculate new row values for entering variables.*

Step VI. *Revise remaining rows.*

Repeat step III to VI till an optimal solution is obtained. This procedure can best be explained with the help of a suitable example.

Example 4.1. *Solve the following linear programming problem by simplex method:*

Maximize $Z = 10X_1 + 20X_2$

Subject to the following constraints:

$$3X_1 + 2X_2 \leq 1200$$
$$2X_1 + 6X_2 \leq 1500$$
$$X_1 \leq 350$$
$$X_2 \leq 200$$
$$X_1, X_2 > 0$$

Solution. Step I. *Formulate the problem*.

Problem is already stated in the mathematical model.

Step II. *Set-up the initial simplex table with the slack variables in solution*. By introducing the slack variables, the equations in step I, *i.e.*, the mathematical model can be rewritten as follows:

$$3X_1 + 2X_2 + S_1 = 1200$$
$$2X_1 + 6X_2 + S_2 = 1500$$
$$X_1 + S_3 = 350$$
$$X_2 + S_4 = 200$$

where S_1, S_2, S_3 and S_4 are the slack variables. Let us rewrite the above equation in a symmetrical manner so that all the four slacks S_1, S_2, S_3 and S_4 appear in all the equations.

$$3X_1 + 2X_2 + 1S_1 + 0S_2 + 0S_3 + 0S_4 = 1200$$
$$2X_1 + 6X_2 + 0S_1 + 1S_2 + 0S_3 + 0S_4 = 1500$$
$$1X_1 + 0X_2 + 0S_1 + 0S_2 + 1S_3 + 0S_4 = 350$$
$$0X_1 + 1X_2 + 0S_1 + 0S_2 + 0S_3 + 1S_4 = 200$$

Let us write the objective function Z by introducing the slacks in it.

$$Z = 10X_1 + 20X_2 + 0S_1 + 0S_2 + 0S_3 + 0S_4$$

The first simplex table can now be written as.

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Table 4.1

C_j	Solution Mix	Rs. 10	Rs. 20	00	0	0	0	Contribution unit quantity
		X_1	X_2	S_1	S_2	S_3	S_4	
0	S_1	3	2	1	0	0	0	1200
0	S_2	2	6	0	1	0	0	1500
0	S_3	1	0	0	0	1	0	350
0	S_4	0	① Key element	0	0	0	1	200 → key row
Z_j		0	0	0	0	0	0	0
$(C_j - Z_j)$		10	20	0	0	0	1	
			↑ key column					

The first simplex table is shown in Table 4.1. The table is explained as below:

1. Row 1 contains C_j or the contribution to total profit with the production of one unit of each product X_1 and X_2. This row gives the coefficients of the variables in the objective function which will remain the same. Under column 1 (C_j) is provided profit per unit of 4 variables S_1, S_2, S_3, S_4 which is zero.

2. All the variables S_1, S_2, S_3, S_4 are listed under solution mix. Their profit is zero and written under column 1 (C_j) as explained above.

3. The constraint variables are written to the right of solution mix. These are X_1, X_2, S_1, S_2, S_3 and S_4. Under these are written coefficient of variables and under each are written the coefficients of particular variable as they appear in the constraint equations. For example, the coefficients X_1, X_2, S_1, S_2, S_3 and S_4 in first constraint equations are 3, 2, 1, 0, 0 and 0, respectively which are written under these variables in the first level. Similarly, the remaining 3 rows represent the coefficients of the variables as they appear in the other 3 constraint equations. The entries in the contribution unit quantity column represent the right hand side of each constraint equation. These values are 1200, 1500, 350 and 200 respectively, for the given problem.

4. The Z_j values in the second row from the bottom refer to the amount of gross profit that is given up by the introducing one unit of that variable into the solution. The subscript j refers to the specific variable being considered. The Z_j value under the quantity column is the total profit for the solution. In the first column all the Z_j values will be zero because no real product is being manufactured and hence there is no gross profit to be lost if they are replaced.

5. The bottom row of the table contains net profit per unit obtained by introducing one unit of a given variable into the solution. This row is designated as the $C_j - Z_j$ row. The procedure for calculating Z_j and $C_j - Z_j$ values is given below :

Calculation of Z_j

$C_j \times X_1$	$C_j \times X_2$	$C_j \times S_1$
$0 \times 3 = 0$	$0 \times 2 = 0$	$0 \times 1 = 0$
+	+	+
$0 \times 2 = 0$	$0 \times 6 = 0$	$0 \times 0 = 0$
+	+	+
$0 \times 1 = 0$	$0 \times 0 = 0$	$0 \times 0 = 0$
+	+	+
$0 \times 0 = 0$	$0 \times 1 = 0$	$0 \times 0 = 0$
$Z_{X_1} = 0$	$Z_{X_2} = 0$	$Z_{X_2} = 0$

Similarly, Z_{S_2}, Z_{S_3} and Z_{S_4}, can be calculated as 0 each.

Calculation of $C_j - Z_j$

$$C_{X_1} - Z_{X_1} = 10 - 0 = 10$$
$$C_{X_2} - Z_{X_2} = 20 - 0 = 20$$
$$C_{S_1} - Z_{S_1} = 0 - 0 = 0$$
$$C_{S_2} - Z_{S_2} = 0 - 0 = 0$$
$$C_{S_3} - Z_{S_3} = 0 - 0 = 0$$
$$C_{S_4} - Z_{S_4} = 0 - 0 = 0$$

The total profit for this solution is Rs. zero.

Step 3. *Determine the variable to be brought into the solution*. An improved solution is possible if there is a positive value in $C_j - Z_j$ row. The variable with the largest positive value in the $C_j - Z_j$ row is subjected as the objective is to maximize the profit. The column associated with this variable is referred to as 'key column' and is designated by a small arrow beneath this column. In the given example, 20 is the largest possible value corresponding to X_2 which is selected as the key column.

Step 4. *Determine which variable is to be replaced*. To make this determination, divide each amount in the contribution quantity column by the amount in the comparable row of key column, X_2 and choose the variable associated with smallest quotient as the one to be replaced. In the given example, these values are calculated as:

for the S_1 row, 1200/2 = 600

for the S_2 row, 1500/6 = 250

for the S_3 row, 350/0 = ∞

for the S_4 row, 200/1 = 200

Since the smallest quotient is 200 corresponding to S_4, S_4 will be replaced, and its row is identified by the small arrow to the right of the table as shown. The quotient represents the maximum value of X which could be brought into the solution.

Step 5. *Calculate the new row values for entering the variable*. The introduction of X_2 into the solution requires that the entire S_4 row be replaced. The values of X_2, the replacing row, are obtained by dividing each value presently in the S_4 row by the value in column X_2 in the same row. This value is termed as the key or the pivotal element since it occurs at the intersection of key row and key column.

NOTES

X_2 ← key column

2

6 key element

0

S_4 0 (1) 0 0 0 1 200 ← key row

20

The row values for entering variable X_2 can be calculated as follows :

$$0/1 = 0;\ 1/1 = 1;\ 0/1 = 0;\ 0/1 = 0;\ 0/1 = 0;\ 1/1 = 1;\ 200/1 = 200$$

Step 6. *Update the remaining rows*. The new S_2 row values are 0, 1, 0, 0, 1 and 200 which are same as the previous table as the key element happens to be 1. The introduction of a new variable into the problem will affect the values of remaining variables and a second set of calculations need to be performed to update the initial table. These calculations are performed as given here:

Updated S_1 row = old S_1 row – intersectional element of old S_1 row × corresponding element of new X_2 row

$$= 3 - [2 \times 0] = 3 = 2 - [2 \times 1] = 0$$
$$= 1 - [2 \times 0] = 1 = 0 - [2 \times 0] = 0$$
$$= 0 - [2 \times 0] = 0 = 0 - [2 \times 1] = -2$$
$$= 1200 - [2 \times 200] = 800$$

Similarly, the updated elements of S_2 and S_3 rows can be calculated as follows :

Elements of updated S_2 row	**Elements of updated S_3 row**
$2 - [6 \times 0] = 2$	$1 - [0 \times 0] = 1$
$6 - [6 \times 1] = 0$	$0 - [0 \times 1] = 0$
$0 - [6 \times 0] = 0$	$0 - [0 \times 0] = 0$
$1 - [6 \times 0] = 1$	$0 - [0 \times 0] = 0$
$0 - [6 \times 0] = 0$	$1 - [0 \times 0] = 1$
$0 - [6 \times 1] = -6$	$0 - [0 \times 1] = 0$
$1500 - [6 \times 200] = 300$	$350 - [0 \times 200] = 350$

Rewriting the second simplex table with the updated elements as shown in Table 4.2:

Table 4.2

C_j	**Solution Mix**	**Rs.10**	**Rs.20**	**0**	**0**	**0**	**0**	**Contribution**		
		X_1	X_2	S_1	S_2	S_3	S_4	**Quantity**	**Ratio**	
0	S_1	3	0	1	0	0	– 2	800	266.7	
0	S_2	2	0	0	1	0	– 6	300	150.	→ key row
0	S_3	1	0	0	0	1	0	350	350.	
20	X_2	0	1	0	0	0	1	200	∞	
	Z_J	0	20	0	0	0	20	4000		
	$(C_J - Z_J)$	10	0	0	0	0	– 20			

↑ key column

The new variable entering the solution would be X_1. It will replace the S_2 row which can be shown as follows :

For the S_1 row, 800/3 = 266·7

For the S_2 row, 300/2 = 150

For the S_3 row, 350/1 = 350

For the S_4 row, 200/0 = ∞

As the quotient 150 corresponding to S_2 row is the minimum, it will be replaced by X_1 in the new solution. The corresponding elements of S_2 row can be calculated as follows :

X_1 key element

3

S_4 (2) 0 0 1 0 –6 300 ← key row

1

0

0

10

↑

key element

New elements of S_2 row to be replaced by X_1 are:

2/2 = 1; 0/2 = 0; 0/2 = 0; 1/2 = 1/2; 0/2 = 0; – 6/2 = – 3; 300/2 = 150;

The updated elements of S_1 and S_3 rows can be calculated as follows :

Elements of updated S_1 row	**Elements of updated S_3 row**	**Elements of updated X_2 row**
3 – [3 × 1] =	0 1 – [1 × 1] =	0 0 – [0 × 1] = 0
0 – [3 × 0] =	0 0 – [1 × 0] =	0 1 – [0 × 0] = 1
1 – [3 × 0] =	1 0 – [1 × 0] =	0 0 – [0 × 0] = 0
0 – [3 × 1/2] =	– 3/2 0 – [1 × 1/2] =	– 1/2 0 – [0 × 1/2] = 0
0 – [3 × 0] =	0 1 – [1 × 0] =	1 0 – [0 × 0] = 0
– 2 – [3 × – 3] =	– 7 0 – [1 × – 3] =	3 1 – [0 × – 3] =1
800 – [3 × 150] =	350 350 – [1 × 150] =	200 200 – [0 × 150] =200

Revised simplex table can now be written as shown in Table 4.3 below :

Table 4.3

C_j	Solution Mix	Rs.10	Rs.20	0	0	0	0	Contribution Per unit Quantity	Minimum Ratio
		X_1	X_2	S_1	S_2	S_3	S_4		
0	S_1	0	0	1	– 3/2	0	7	350	50 → key row
10	X_1	1	0	0	1/2	0	– 3	150	–50
0	S_3	0	0	0	– 1/2	1	30	200	66.7
20	X_2	0	1	0	0	0	1	200	200
	Z_J	10	20	0	5	0	– 10	5500	
	$(C_J – Z_J)$	0	0	0	– 5	0	10		
							↑ key column		

Now, the new entering variable will be S_4 and it will replace S_1 as shown below :

350/7 = 50

150/– 3 = – 50

$$200/3 = 66{\cdot}7$$
$$200/1 = 200$$

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In these figures, 50 represent the minimum quotient which corresponds to row S_1. The negative sign is not considered.

The new elements of S_1 row to be replaced by S_4 can be calculated as follows :

S_4

S_1 0   0   1   –3/2   0   (7)   350 ⟵ Key row

Key element

– 3

3

1

–10

10

↑

key column

The new elements of S_4 row would be

$0/7 = 0$; $0/7 = 0$; $1/7 = 1/7$; $(-3/2) \times (1/7) = -3/14$; $0/7 = 0$; $7/7 = 1$; $350/7 = 50$

The updated elements of the other rows can be calculated as follows :

Elements of updated X_1 row	**Elements of updated S_3 row**	**Elements of updated X_2 row**
$1 - [-3 \times 0] = 1$	$0 - [3 \times 0] = 0$	$0 - [1 \times 0] = 0$
$0 - [-3 \times 0] = 0$	$0 - [3 \times 0] = 0$	$1 - [1 \times 0] = 1$
$0 - [-3 \times 1/7] = 3/7$	$0 - [3 \times 1/7] = -3/7$	$0 - [1 \times 1/7] = -1/7$
$1/2 - [-3 \times 3/14] = -1/7$	$-1/2 - [3 \times -3/14] = 1/7$	$0 - [1 \times -3/14] = 3/14$
$0 - [-3 \times 0] = 0$	$1 - [3 \times 0] = 1$	$0 - [1 \times 0] = 0$
$-3 - [-3 \times 1] = 0$	$3 - [3 \times -1] = 0$	$1 - [1 \times 1] = 0$
$150 - [-3 \times 50] = 300$	$200 - [3 \times 50] = 50$	$200 - [1 \times 50] = 150$

The new simplex table can now be written as shown in Table 4.4.

Table 4.4

C_j	Solution Mix	Rs.10	Rs.20	0	0	0	0	Contribution Quantity
		X_1	X_2	S_1	S_2	S_3	S_4	
0	S_4	0	0	1/7	– 3/4	0	1	50
10	X_1	1	0	3/7	– 1/7	0	0	300
0	S_3	0	0	–3/7	1/7	1	0	50
20	X_2	0	1	–1/7	3/14	0	0	150
	Z_J	10	20	10/7	40/14	0	0	6000
	$(C_J - Z_J)$	0	0	–10/7	– 40/14	0	0	

As there is no positive value in $C_j - Z_j$ row it represents the optimal solution, which is given as :

$$X_1 = 300 \text{ units} : \quad X_2 = 150 \text{ units}$$

And the maximum profit Z = Rs 6000

Minimization Problems

An identical procedure is followed for solving the minimization problems. Since the objective is to minimize rather than maximize, a negative ($C_j - Z_j$) value indicates potential improvement. Therefore, the variable associated with largest negative ($C_j - Z_j$) value would be brought into the solution first. Additional variables are brought into set-up such problems. However, such problems involve greater than or equal to constraints, which need to be treated separately from less than or equal to constraints, which are typical of maximization problems. In order to convert such inequalities, the following procedure may be adopted:

For example, if the constraint equation is represented as :

$$3X_1 + 2X_2 > 1200$$

To convert this into equality, it would be written as :

$$3X_1 + 2X_2 - S_1 = 1200$$

where S_1 is a slack variable. However, this will create a difficulty in the simplex method because of the fact that the initial simplex solution starts with slack variables and a negative value ($-1S_1$) would be in the solution, a condition which is not permitted in linear programming. To overcome this problem, the simplex procedure requires that another variable known as artificial variable be added to each equation in which a slack variable is subtracted. An artificial variable may be thought of as representing a fictitious product having very high cost which though permitted in the initial solution to a simplex problem, would never appear in the final solution. Defining A as an artificial variable, the constraint equation can be written as :

$$3X_1 + 2X_2 - 1S_1 + 1A_1 = 1200$$

Assuming the objective function is to minimize cost, it would be written as :

$$10X_1 + 20X_2 + 0S_1 + MA_1 \text{ to be minimized.}$$

where M is assumed to be very large cost (say, 1 million). Also S_1 is added to the objective function even though it is negative in constraint equation. An artificial variable is also added to constraint equations with equality signs, *e.g.*, if the constraint equation is

$$3X_1 + 2X_2 = 1200$$

then, in simplex it would change to

$$3X_1 + 2X_2 + 1A_1 = 1200$$

to satisfy simplex requirement and would be reflected as MA in the objective function.

Example 4.2. *ABC company manufactures and sells two products P_1 and P_2. Each unit of P_1 requires 2 hours of machining and 1 hour of skilled labour. Each unit of P_2 requires 1 hour of machining and 2 hours of labour. The machine capacity is limited to 600 machine hours and skilled labour is limited to 650 man hours. Only 300 units of product P_2 can be sold in the market. You are required to:*

(i) *develop a suitable model to determine the optimal product mix;*

(ii) *find out the optimal product mix and the maximum contribution. Unit contribution from product P_1 is Rs. 8 and from product P_2 is Rs. 12;*

(iii) *determine the incremental contribution/unit of machine-hours, per unit of labour and per unit of product P_1.*

Solution.

Step 1. *Formulation of LP model.*

Let X_1 and X_2 be the number of units to be manufactured of the two products P_1 and P_2 respectively. We are required to find out the number of units of the two products to be

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manufactured to maximize contribution, *i.e.*, profits when individual contribution of the two products are given. LP model can be formulated as follows :

Maximize $Z = 8X_1 + 12X_2$

Subject to conditions/constraints

$$2X_1 + X_2 \le 600 \text{ (Machine time constraint)}$$
$$X_1 + 2X_2 \le 650 \text{ (Labour-time constraint)}$$
$$X_2 \le 300 \text{ (Marketing constraint of product } P_2)$$

Step 2. *Converting constraints into equations.*

LP problem has to be written in a standard form, for which the inequalities of the constraints have to be converted into equations. For this purpose, we add a slack variable to each constraint equation. Slack is the unused or spare capacity for the constraints to which it is added. In less than (£) type of constraint, the slack variable denoted by S, is added to convert inequalities into equations. S is always a non-negative figure or 0. If S is negative, it may be seen that the capacity utilised will exceed the total capacity, which is absurd. The above inequalities of this problem can be rewritten by adding suitable slack variables and converted into equations as follows:

$$2X_1 + X_2 + S_1 = 600$$
$$X_1 + 2X_2 + S_2 = 650$$
$$X_2 + S_3 = 300$$
$$X_1, X_2, S_1, S_2, S_3 > 0$$

Slack variables S_1, S_2 and S_3 contribute zero to the objective function since they represent only unused resources. Let us include these slack variables in the objective function. Then maximize

$$Z = 8X_1 + 12X_2 + 0S_1 + 0S_2 + 0S_3$$

Step 3. *Set-up the initial solution.*

Let us recollect that the computational procedure in the simplex method is based on the following fundamental property:

"The optimal solution to a Linear Programming problem always occurs at one of three corner points of the feasible solution space".

It means that the corner points of the feasible solution region can provide the optimal solution. Let the search start with the origin which means nothing is produced at origin (0, 0) and the value of decision variable X_1 and X_2 is zero. In such a case, $S_1 = 600$, $S_2 = 650$, $S_3 = 300$ are the spare capacities as nothing (0) is being produced. In the solution at origin we have two variables X_1 and X_2 with zero value and three variables (S_1, S_2 and S_3) with specific values of 600, 650 and 300. The variables with non-zero values, *i.e.*, S_1, S_2 and S_3 are called the **basic variables** whereas the other variables with zero values *i.e.* X_1, X_2 and X_3 are called **non-basic variables**. It can be seen that the number of basic variables is the same as the number of constraints equations (three in the present problem). The solution with basic variables is called **basic solution** which can be further divided into **Basic Feasible Solution** and **Basic Infeasible Solution.** The first type of solutions are those which satisfy all the constraints. In Simplex Method, we search for basic feasible solution only.

Step 4. *Developing initial simplex table.*

The initial decision can be put in the form of a table which is called a *Simplex Table* or *Simplex Matrix*. The details of the matrix are as follows:

1. Row 1 contains C_j or the contribution to total profit with the production of one unit of each product P_1 and P_2. Under column 1 (C_j) are listed the profit coefficients of the basic variables. In the present problem, the profit coefficients of S_1, S_2 and S_3 are zero.

2. In the column labelled Solution Mix or Product Mix are listed the variables S_1, S_2 and S_3, their profits are zero and written under column 1 (C_j) as explained above.

3. In the column labelled 'contribution unit quantity' are listed the values of basic variables included in the solution. We have seen in the initial solution $S_1 = 600$, $S_2 = 650$ and $S_3 = 300$. These values are listed under this column on the right side as shown in Table 4.5. Any variables not listed under the solution-mix column are the non-basic variables and their values are zero.

4. The total profit contribution can be calculated by multiplying the entries in column C_j and column 'contribution per unit quantity' and adding them up. The total profit contribution in the present case is $600 \times 0 + 650 \times 0 + 300 \times 0 = 0$.

5. Numbers under X_1 and X_2 are the physical ratio of substitution. For example, number 2 under X_1, gives the ratio of substitution between X_1 and S_1. In simple words, if we wish to produce 2 units of product P_1, *i.e.*, X_1, 2 units of S_1 must be sacrificed. Other numbers have similar interpretation. Similarly, the number in the 'identity matrix' columns S_1, S_2 and S_3 can be interpreted as ratios of exchange. Hence the numbers under the column S_1, represents the ratio of exchange between S_1 and the basic variables S_1, S_2 and S_3.

6. Z_j and $C_j - Z_j$ are the two final rows. These two rows provide us the total profit and help us in finding out whether the solution is optimal or not Z_j and $C_j - Z_j$ can be found out in the following manner:

   (a) $Z_j = C_j$ of S_1 (0) × coefficients of X_1 in S_1 row (2) + C_j of S_2 (0) × coefficients of X_1 in S_2 row (1) + C_j of S_3 (0) × coefficient X_1 in S_3 row (1) = $0 \times 2 + 0 \times 1 + 0 \times 1 = 0$

   Using the same procedure Zj for all the other variable columns can be worked out as shown in the completed first Simplex table given in Table 4.5.

   (b) The number in the ($C_j - Z_j$) row represent the net profit that will result from introducing 1 unit of each product or variable into the solution. This can be worked out by subtracting, Z_j total for each column from the C_j values at the top of that variable's column. For example, $C_j - Z_j$ number in the X_1 column will $8 - 0 = 8$, in the X_2 column it will be $12 - 0 = 12$, etc.

7. The value of the objective function can be obtained by multiplying the elements in C_j column with the corresponding elements in the C_j rows, i.e., in the present case $Z = 8 \times 0 + 12 \times\ = 0$

Table 4.5

C_j	Solution Mix	8	12	0	0	0	Contribution Per unit Quantity
		X_1	X_2	S_1	S_2	S_3	
0	S_1	2	1	1	0	0	600
0	S_2	1	2	0	1	0	650
0	S_3	0	1	0	0	1	300
	S_4	0	0	0	0	0	
	$(C_J - Z_J)$	8	12	0	0	0	

NOTES

8. By examining the number in the $(C_j - Z_j)$ row, we can see that total profit can be increased by Rs 8 for each unit of product X_1 added to the product mix or by Rs 12 for each unit of product X_2 added to the product mix. A positive $(C_j - Z_j)$ indicates that profits can still be improved. A negative number of $(C_j - Z_j)$ would indicate the amount by which the profits would decrease, if one unit of the variable was added to the solution. Hence, optimal solution is reached only when there are no positive numbers in $(C_j - Z_j)$ row.

Step 5. *Test for optimally.*

Now, we must test whether the results obtained are optimal or it is possible to carryout any improvements. It can be done in the following manner:

1. Selecting the entering variable. We have to select which of the variables, out of the two non-basic variables X_1 and X_2, will enter the solution. We select the one with maximum value of $C_j - Z_j$. Variable X_1 has a $(C_j - Z_j)$ value of 8 and X_2 has a $(C_j - Z_j)$ value of 12. Hence, we select variable X_2 as the variable to enter the solution mix and identify the column in which it occurs as the key column with the help of a small arrow.
2. Selecting the variable that leaves the solution. As a variable is entering the solution, we have to select a variable which will leave the solution. This can be done as follows :
   (a) Divide each number in the solution value or contribution unit quantity column by corresponding number in the key column, i.e., divide 600, 650 and 300 by 1, 0

Table 4.6

C_j	Solution Mix	8 X_1	12 X_2	0 S_1	0 S_2	0 S_3	Solution values	Minimum ratio	
0	S_1	2	1	1	0	0	600	600	
0	S_2	1	(2)	0	1	0	650	325	→ key row
0	S_3	0	1	0	0	1	300	300	
	Z_4	0	0	0	0	0			
	$(C_J - Z_J)$	8	12	0	0	0			
			↑ key column						

   (b) Select the row with smallest non-negative ratio as the row to be replaced, in present example the ratios are:

   For S_1 row, 600/1 = 600 unit of X_2

   For S_2 row, 650/2 = 325 units of X_2

   For S_3 row, 300/1 = 300 units of X_2

   It is clear that S_2 (with minimum ratio) is the departing variable. This row is called the key row.

   (c) The number at the intersection of key row and key column is called the key number which is 2 in the present case and is circled in the table.

Step 6. *Developing second simplex table.*

Now, we can develop the second simplex table.

NOTES

TABLE 4.7 Second Simplex Table

C_j	Solution Mix	8	12	0	0	0	Solution values	Minimum ratio
		X_1	X_2	S_1	S_2	S_3		
0	S_1	2	0	1	0	0	300	150
0	S_2	1	0	0	1	0	50	50 → key row
12	S_3	0	1	0	0	1	300	∞
	Z_4	0	12	0	0	0	3600	
	$(C_J - Z_J)$	8 ↑ key column	0	0	0	0		

Now, X_1 will be replaced.

TABLE 4.8 Third Simplex Table

C_j	Solution Mix	8	12	0	0	0	Solution values
		X_1	X_2	S_1	S_2	S_3	
0	S_1	2	0	1	0	0	200
8	X_1	1	0	0	1	0	50
12	X_2	0	1	0	0	1	300
	Z_4	8	12	0	0	0	4000
	$(C_J - Z_J)$	0	0	0	0	0	

As R_2 will remain same because pivot element is already unity.

Maximum Profit = 4000

We find that the value of objective function has been improved from 0 to ∞. But the correct solution is not optimal as there are positive values (12) and (8) in the $(C_j - Z_j)$ row. Also, since minimum ratio is 325, we select X_2 row to leave the solution as X_2 (key column) will enter the solution. The new X_2 (key) row will remain same as its elements 1/2, 1, 0, 1/2, 0 and 325 have to be divided by key element, i.e., (shown circled in the above Table). However, row S_1 and S_3 elements will undergo change.

Row S_1 = old row number – [corresponding number in key row] × [corresponding fixed ratio]

Fixed ratio = old row number in key column/key number = 0

It can be concluded that this problem does not have a optimal solution as X_2 row is to be replaced by X_2 row.

Example 4.3. *Solve the following problem using Simplex method*

$$Z_{max} = 15x_1 + 2x_2 + 3x_3$$

Subject to $2x_1 + 2x_2 - 4x_3 \le 18$

$8x_1 + 2x_2 + 2x_3 \le 36$

$x_1, x_2, x_3 \ge 0.$

Solution. Converting inequalities into equalities condition are

$2x_1 + 2x_2 - 4x_3 + 0S_1 + 0S_2 = 18$

$8x_1 + 2x_2 + 2x_3 + 0S_1 + 0S_2 = 36$

$x_1, x_2, x_3, S_1, S_2 = 0$

Constructing the first simplex table

NOTES

TABLE 4.9

C_j	Solution Mix	5	2	3	0	0	Solution values	Minimum ratio
		X_1	X_2	X_3	S_1	S_2		
0	S_1	2	2	– 4	1	0	18	9
0	S_2	8	2	2	0	1	36	4.5
Z_j		0	0	0	0	0		
$C_J – Z_J$		–15	–2	– 3	0	0		

Since – 15 is least in $C_j – Z_j$, S_2 will be replaced by X_1.

TABLE 4.10

C_j	Solution Mix	15	2	3	0	0	Solution values
		x_1	x_2	x_3	S_1	S_2	
0	S_1	0	$\frac{3}{2}$	$\frac{-9}{2}$	1	$\frac{-1}{4}$	9
15	X_1	1	$\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{1}{8}$	4.5
Z_j		15	$\frac{15}{4}$	$\frac{15}{4}$	0	$\frac{15}{8}$	67.5
$C_J – Z_J$		0	$\frac{7}{4}$	$\frac{3}{4}$	0	$\frac{5}{2}$	

Elements of second row (x_1)

Divide every element of second row by 8 of previous table

Elements of second row= Elements of first row by previous table

– Elements of second row in previous table × conversion factor

$$18 – 36 \times \frac{2}{8} = 9$$

$$2 – 8 \times \frac{2}{8} = 0$$

$$2 – 2 \times \frac{2}{8} = \frac{3}{2}$$

$$– 4 – 2 \times \frac{2}{8} = -\frac{9}{2}$$

$$1 – 0 \times \frac{2}{8} = -\frac{1}{4}$$

$$0 – 1 \times \frac{2}{8} = -\frac{1}{4}$$

In Table 4.10 all elements in $Z_j – C_j$ row are positive.

So, optimum solution has been achieved.

$$Z_{max} = 67.5 \text{ where } x_1 = 4.5, x_2 = 0$$

4.3 MINIMIZATION PROBLEMS (ALL CONSTRAINTS OF THE TYPE ≥) BIG 'M' METHOD

We have till now seen in this chapter, the type of problems where profit had to be maximized and the constraints were of the type ≤. However, there could be problems where the objective function has to be minimized (like the availability of funds, raw material or the costs of operations have to be minimized) and the constraints involved may be of the type ≥ or = . In such cases, the simplex method is somewhat different and is discussed in the form of following steps:

Step 1. *Formulation of mathematical model.*

Minimize $Z = C_1 x_1 + C_2 x_2 + C_3 x_3 + \ldots\ldots\ldots\ldots + C_n x_n$

Subject to the constraints

$$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 + \ldots\ldots\ldots a_{1n} x_n \geq b_1$$
$$a_{21} x_1 + a_{22} x_2 + a_{23} x_3 + \ldots\ldots\ldots a_{2n} x_3 \geq b_2$$
$$\vdots \quad \vdots \qquad \vdots \qquad\qquad \vdots \quad \vdots$$
$$a_{m_1} x_1 + a_{m_2} x_2 + a_{m_3} x_3 + \ldots\ldots\ldots a_{mn} x_n \geq b_m$$

where $x_1, x_2, x_3 \ldots\ldots\ldots x_n \geq 0$

Now, we subtract the surplus variables $S_1, S_2, \ldots\ldots\ldots\ldots S_n$, etc., to convert the inequalities into equations.

i.e., Minimize

$$Z = C_1 x_1 + C_2 x_2 + C_3 x_3 + \ldots\ldots\ldots + C_n X_n + 0S_1 + 0S_2 + \ldots\ldots\ldots + 0S_n$$

Subject to the constraints

$$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 + \ldots\ldots\ldots a_{1n} x_n - S_1 = b_1$$
$$a_{21} x_1 + a_{22} x_2 + a_{23} x_3 + \ldots\ldots\ldots a_{2n} x_3 - S_2 = b_2$$
$$\vdots \qquad \vdots \qquad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots$$
$$a_{m_1} x_1 + a_{m_2} x_2 + a_{m_3} x_3 + \ldots\ldots\ldots + a_{mn} x_n - S_m = b_m$$

where $x_i \geq 0 \ (i = 1, 2, \ldots\ldots\ldots n)$

$S_j \geq 0 \ (j = 1, 2, \ldots\ldots\ldots m)$

As in the maximization problem, initial basic solution is obtained by putting $x_1 = x_2 = \ldots\ldots\ldots = x_n = 0$

So,

$$-S_1 = b_1 \quad \text{or} \quad S_1 = -b_1$$
$$-S_2 = b_2 \quad \text{or} \quad S_2 = -b_2$$
$$\vdots \quad \vdots \qquad \vdots \quad \vdots$$
$$-S_m = b_m \text{ or } S_m = -b_m$$

It may be seen that $S_1, S_2 \ldots\ldots\ldots S_m$ being negative violate the non-negativity constraint and hence is not feasible. Hence, in the system of constraints we introduce *m* new variables $A_1, A_2 \ldots\ldots\ldots A_m$ known as artificial variable. By introducing these variables the equations are:

$$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 + \ldots\ldots\ldots + a_{1n} x_n - S_1 + A_1 = b_1$$
$$a_{21} x_1 + a_{22} x_2 + a_{23} x_3 + \ldots\ldots\ldots + a_{2n} x_n - S_2 + A_2 = b_2$$
$$\vdots \qquad \vdots \qquad \vdots \qquad\qquad \vdots \quad \vdots$$
$$a_{m_1} x_1 + a_{m_2} x_2 + a_{m_3} x_3 \ldots\ldots\ldots + a_{mn} x_n - S_m + A_m = b_m$$

where $x_j \geq 0$ $(i = 1, 2, 3, \ldots\ldots\ldots n)$

$S_j \geq 0$ $(j = 1, 2, 3, \ldots\ldots m)$

$A_j \geq 0$ $(j = 1, 2, 3, \ldots\ldots m)$

As we have introduced artificial variables $A_1, A_2 \ldots\ldots\ldots A_m$ this has to be taken out of the solution. For this purpose, we introduce a very large value (M) assigned to each of artificial variable and zero to each of the surplus variables as the coefficient values in the objective function. The problem now becomes

Minimize

$$Z = C_1 x_1 + C_2 x_2 + C_3 x_3 + \ldots\ldots\ldots + C_n x_n + 0S_1 + 0S_2 + \ldots\ldots + 0S_m + MA_1 + MA_2 + \ldots\ldots\ldots + MA_m$$

Subject to the constraints

$$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 + \ldots\ldots\ldots a_{1n} x_n - S_1 + A_1 = b_1$$
$$a_{21} x_1 + a_{22} x_2 + a_{23} x_3 + \ldots\ldots\ldots a_{2n} x_3 - S_2 + A_2 = b_2$$
$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$
$$a_{m_1} x_1 + a_{m_2} x_2 + a_{m_3} x_3 + \ldots\ldots\ldots a_{mn} x_n - S_m + A_m = b_m$$

Step 2. *Setting up of initial simplex table*

Here, we allot 0 values to variables $x_1 = x_2 = x_3 = \ldots\ldots = x_n = 0$ so that $A_1 = b_1$, $A_2 = b_2 \ldots\ldots\ldots A_m = b_m$.

TABLE 4.11 Initial Simplex Table

		C_j	$C_1\ C_2\ C_3 \ldots\ldots C_n\ 0\ \ 0\ M \ldots\ldots M$	Minimum ration
CB	Solution Mix	Solution Values	$x_1\ x_2\ x_3 \ldots\ldots x_n\ \ S_1\ S_2\ S_m\ \ A_1\ A_2\ A_m$	
CB_1	A_1	b_1	$a_{11}\ \ a_{12} \ldots\ldots a_{1n}\ \ -1\ \ 0\ \ 0\ \ 1\ \ 0 \ldots\ldots 0$	
CB_2	A_2	b_2	$a_{21}\ \ a_{22} \ldots\ldots a_{2n}\ \ 0\ \ -1\ \ 0\ \ 1\ \ 0 \ldots\ldots 0$	
$\vdots$	$\vdots$	$\vdots$	$\vdots\ \ \vdots\ \ \vdots\ \ \vdots\ \ \vdots\ \ \vdots\ \ \vdots\ \ \vdots\ \ \vdots\ \ \vdots$	
CB_n	A_n	b_m	$a_{m_1}\ \ a_m \ldots\ldots a_{mn}\ \ 0\ \ 0\ \ -1\ \ 0\ \ 0\ \ \ 1$	
Z_j			$0\ \ 0\ \ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0 \ldots\ldots 0$	
$(C_j - Z_j)$			$C_1\ \ C_2 \ldots\ldots C_n\ \ 0\ \ 0\ \ 0\ \ M\ \ M \ldots\ldots M$	

Step 3. *Test for optimality*

Calculate the elements of $(C_j - Z_j)$ row.

(*a*) If all $(C_j - Z_j) \geq 0$ then the basic feasible solution is optimal.

(*b*) If any one $(C_j - Z_j) \leq 0$ then pick up the largest negative number in this row. This is the key column and determines the variable entering the solution.

Now, the second simplex table can be constructed.

Step 4. *Test for feasibility*

Determine the key row and key number (element) in the same manner as is done in the maximization problem.

Example 4.4. *A special diet for a patient in the hospital must have at least 8000 units of vitamins, 100 units of minerals and 2800 units of calories. Two types of foods X and Y are available in the market at the cost of Rs. 8 and Rs. 6 respectively. One unit of X contains 400 units of vitamins, 2 units of minerals and 80 units of calories. One unit of food B contains 200 units of vitamins, 4 units of minerals and 80 units of calories. What combination of foods X and Y be used so that the minimum requirement of vitamins, minerals and calories is maintained and the cost incurred by the hospital is minimized ? Use simplex method.*

Solution. Mathematical model of the problem is as follows :

Minimize $Z = 8x_1 + 6x_2$

Subject to the constraints

$$400x_1 + 200x_2 \geq 8000 \text{ (Constraint of minimum vitamins)}$$
$$2x_1 + 4x_2 \geq 100 \text{ (Constraint of minimum minerals)}$$
$$80x_1 + 80x_2 \geq 2800 \text{ (Constraint of minimum calories)}$$
$$x_1, x_2 \geq 0 \text{ (Non-negativity constraint)}$$

where x_1 and x_2 are the number of units of food X and food Y. Now, the constraint inequalities can be converted into equations. Here, we take an initial solution with very high cost, as opposed to the maximization problem where we had started with an initial solution with no profit. We subtract surplus variables S_1, S_2 and S_3.

$$400x_1 + 200x_2 - S_1 = 8000$$
$$2x_1 + 4x_2 - S_2 = 100$$
$$80x_1 + 80x_2 - S_3 = 2800$$

The surplus variables S_1, S_2 and S_3 introduced in these equations represent the extra unit of vitamins, minerals and calories over 8000 units, 100 units and 2800 units in the least cost combination.

Let x_1, x_2 be zero in the initial solution.

Hence $S_1 = -8000$

$S_2 = -100$

$S_3 = -2800$

This is not feasible as S_1, S_2 and $S_3 \geq 0$ and cannot be negative. We have to see that S_1, S_2 and S_3 do not appear (as they are – ve) in the initial solution. So, if x_1, x_2 and S_1, S_2, S_3 are all zero, new foods which can substitute food X and Y must be introduced. A_1, A_2 and A_3 are the artificial variables to be introduced. Let the artificial variables (foods) be of are very large price, M per unit

$$400x_1 + 200x_2 - S_1 + A_1 = 8000$$
$$2x_1 + 4x_2 - S_2 + A_2 = 100$$
$$80x_1 + 80x_2 - S_3 + A_3 = 2800$$

and Z objective function

Minimize $Z = 8x_1 + 6x_2 + 0S_1 + 0S_2 + 0S_3 + MA_1 + MA_2 + MA_3$

where $x_1, x_2, S_1, S_2, S_3, A_1, A_2, A_3 \geq 0$

Now, it is possible to set-up initial solution by putting $x_1 = x_2 = S_1 = S_2 = S_3 = 0$ in such a manner that $A_1 = 8000$, $A_2 = 100$ and $A_3 = 2800$.

NOTES

TABLE 4.12 First Simplex Table

		C_j	8	6	0	0	0	M	M	M	
C_B	B Solution mix Variables	b $(= x_B)$ Solution values	x_1	x_2	S_1	S_2	S_3	A_1	A_2	A_3	Minimum ratio
M	A_1	8000	400	200	–1	0	0	1	0	0	20 → key row
M	A_2	100	2	4	0	–1	0	0	1	0	50
M	A_3	2800	80	80	0	0	–1	0	0	1	35
Z_j			482 M	284 M	– M	– M	– M	M	M	M	
$(C_j - Z_j)$			8 – 482M	6 – 284M	M	M	M	0	0	0	

↑ key column (x_1)

x_1 is the key column entering the solution, A is the departing row and 400 (circled) in the table is the key number (element).

Now, apply the row operations.

(*i*) R – 1 (new) → $\frac{1}{400}$ R – 1 (old)

(*ii*) R – 2 (new) → R – 2 (old) – 2R – 1 (new)

(*iii*) R – 3 (new) → R – 3 (old) – 80 R – 1 (new)

TABLE 4. 13 Second Simplex Table

		C_j	8	6	0	0	0	M	M	M	
C_B	Solution mix Variables (= B)	Solution values $b(= X_B)$	x_1	x_2	S_1	S_2	S_3	A_1	A_2	A_3	Minimum ratio
8	x_1	20	1	$\frac{1}{2}$	$-\frac{1}{400}$	0	0		0	0	40
M	A_2	60	0	3	$\frac{1}{200}$	–1	0		1	0	20 → key row
M	A_3	1200	0	40	$\frac{1}{4}$	0	–1		0	1	30
Z_j			8	4 + 43 M	– 4 + 41 M/200	– M	– M		M	M	
$(C_j - Z_j)$			0	2 – 43 M	4 – 41 M/200	M	M			0	

↑ key column (x_2)

Value of Z calculated as follows:

$$Z_j(x_1) = 8 \times 1 \;\; M \times 0 = 8$$

$$Z_j(x_2) = \frac{1}{2} \times 8 + 3 \times M + 40\text{ M} = 4 + 43\text{ M}$$

$$Z_j(S_1) = \frac{1}{400} \times 8 + \frac{1}{200}\text{ M} + \frac{1}{5}\text{ M} = \frac{-4 + 41\text{M}}{200}$$

$Z_j\ (S_2) = -M$

$Z_j\ (S_3) = -M$

$Z_j\ (A_2) = M;\ Z_j\ (A_3) = M$

NOTES

It is clear from the above table, that x_2 enters the solution and A_2 departs, using the following row operations:

We introduce x_2 and remove A_2.

(*i*) R – 2 (new) $\rightarrow \frac{1}{3}$ R – 2 (new)

(*ii*) R – 1(new) $\rightarrow$ R – 1 (old) – $\frac{1}{2}$ R – 2 (new)

(*iii*) R – 3 (new) $\rightarrow$ R – 3 (old) – 40 R – 2 (new)

$$R - 2\ (\text{new}) = 20,\ 0,\ 1,\ \frac{1}{600}, -\frac{1}{3}, 0, \frac{1}{3}, 0$$

$$R - 1\ (\text{new}) = 10,\ 1,\ 0,\ -\frac{1}{300}, \frac{1}{6}, 0$$

$$R - 3\ (\text{new}) = 400,\ 0,\ 0, \frac{2}{15}, \frac{40}{3}, -1$$

Now, the third simplex table can be drawn.

TABLE 4.14 Third Simplex Table

		C_j	**8**	**6**	**0**	**0**	**0**	**M**	**M**	**M**	
C_B	**Solution mix Variables (= B)**	**Solution values b (= x_B)**	**x_1**	**x_2**	**S_1**	**S_2**	**S_3**	**A_1**	**A_2**	**A_3**	**Minimum ratio**
8	x_1	10	1	0	$-\frac{1}{300}$	$\frac{1}{6}$	0	–	–	0	60
6	x_2	20	0	1	$\frac{1}{600}$	$-\frac{1}{3}$	0	–	–	0	– 60
M	A_3	400	0	0	$\frac{2}{5}$	$\frac{40}{3}$	–1	–	–	1	30
Z_j			8	6	$\frac{-1+8M}{60}$	$\frac{-2+4M}{3}$	– M	–	–	M	
$(C_j - Z_j)$			0	0	$\frac{1+8M}{60}$	$\frac{2+4M}{3}$	M	–	–	0	

→ key row

↑ key column

It can be seen S_2 has to be introduced and A_3 has to depart. This procedure can be adopted for further improving the solution by constructing fourth simplex table and so on.

4.4 MINIMIZING CASE—CONSTRAINTS OF MIXED TYPE (£ AND ≥)

We have seen the examples earlier where the constraints were either ≥ type or £ type. Both there are problems where the constraint equation could contain both types of constraints. This type of problem is illustrated with the help of an example.

NOTES

Example 4.5. *A metal alloy used in manufacture of rifles uses two ingredients A and B. A total of 120 units of alloy is used for production. Not more than 60 units of A can be used and at least 40 units of ingredient B must be used in the alloy. Ingredient A costs Rs. 4 per unit and ingredient B costs Rs. 6 per unit. The company manufacturing rifles is keen to minimize its costs. Determine how much of A and B should be used.*

Solution. Mathematical formulation of the problem is

Minimize cost $Z = 4x_1 + 6x_2$

Subject to constraints

$x_1 + x_2 = 120$ (Total units of alloy)

$x_1 \le 60$ (Ingredient A constraint)

$x_2 \ge 40$ (Ingredient B constraint)

$x_1, x_2 \ge 0$ (Non-negativity constraint)

where x_1 and x_2 number of units of ingredient A and B respectively. Let x_1 and $x_2 = 0$ and let us introduce an artificial variable which represents a new ingredient with very high cost M.

$x_1 + x_2 + A_1 = 120$

Also $x_1 + S_1 = 60$

Third constraint $x_2 - S_2 + A_2 = 40$

Now, the standard form of the problem is

Minimize $Z = 4x_1 + 6x_2 + MA_1 + 0S_1 + 0S_2 + MA_2$

Subject to the constraints

$$x_1 + x_2 + A_1 = 120$$
$$x_1 + S_1 = 60$$
$$x_2 - S_2 + A_2 = 40$$
$$x_1, x_2, S_1, S_2, A_1, A_2 \ge 0$$

Initial basic solution is obtained by putting $x_1 = x_2 = 0$ and $S_1 = S_2 = 0$ so that $A_1 = 100$, $S_1 = 60$, $A_2 = 40$.

TABLE 4.15 First Simplex Table

		C_j	4	6	M	0	0	M	Minimum ratio
C_B	Solution mix	Solution values	x_1	x_2	A_1	S_1	S_2	A_2	
M	A_1	120	1	1	1	0	0	0	120
0	S_1	60	1	0	0	1	0	0	–
M	A_2	40	0	1	0	0	–1	0	40 → key row
Z_j			M	2M	M	0	–M	M	
$(C_j - Z_j)$			4 – M	6 – 2M ↑ key column	0	0	M	0	

6 – 2M is the largest negative number hence, x_2 will enter the solution and since 40 is the minimum ratio A_2 will depart.

R – 3 (New) → R – 3 (old) as key element is 1.

R – 1 (New) → R – 1 (old) – R – 3 (New)

NOTES

TABLE 4.16 Second Simplex Table

		C_j	4	6	M	0	0	M	Minimum ratio
C_B	Solution mix	Solution values	x_1	x_2	A_1	S_1	S_2	A_2	
M	A_1	80	1	0	1	0	1		80
0	S_1	60	1	0	0	1	0		60 → key row
6	x_2	40	0	1	0	0	–1		–
Z_j			M	6	M	0	M – 6		
$(C_j – Z_j)$			4 – M	0	0	0	–M – 6		

↑ key column

R – 1 (new) = 1 – 0 = 1 ; 1 – 1 = 0, 1 – 0 = 1, 0 – 0 = 0, 0 – (- 1) = 1

i.e., 0, 1, 1, 0, 1, 100 – 40 = 60

x_1 will be introduced and S_1 will depart.

Use the following row operations :

(*i*) R – 2 (new) → R_2 (old)

(*ii*) R – 1 (new) → R_1 (old) – R_2 (new)

R – 2 (new) = 1, 0, 0, 1, 0

R – 1 (new) = 1 – 1 = 0, 0 – 0 = 0, 1 – 0 = 1, 0 – 1 = – 1, 1 – 0 = 1

i.e., 0, 0, 1, – 1, 1

TABLE 4.17 Third Simplex Table

		C_j	4	6	M	0	0	M	Minimum ratio
C_B	Solution mix	Solution values	x_1	x_1	A_1	S_1	S_2	A_2	
M	A_1	40	0	0	1	–1	1		40 → key row
4	x_1	60	1	0	0	1	0		–
6	x_2	40	0	1	0	0	–1		40
Z_j			4	6	M	– M + 4	M + 4		
$(C_j – Z_j)$			0	0	0	M – 4	– M + 6		

↑ key column

We now introduce S_2 and take out A_1 using following row operations:

R – 1 (new) → R – 1 (old)

R – 3 (new) → R – 3 (old) + R – 1 (new)

NOTES

TABLE 4.18 Fourth Simplex Table

		C_j	4	6	M	0	0	M
C_B	Solution mix	Solution values	x_1	x_1	A_1	S_1	S_2	A_2
0	S_1	40	0	0		–1	1	
4	x_1	60	1	0		1	0	
6	x_2	80	0	1		– 1	0	
Z_j			4	6	–	– 2	0	
$(C_j – Z_j)$			0	0	–	2	0	

Since all the numbers in $(C_j – Z_j)$ are either zero or positive, this is the optimal solution.

$x_1 = 60$, $x_2 = 80$ and $Z = 40 \times 60 + 6 \times 80 = \text{Rs. } 720$

Maximization Case-constraints of Mixed Type

A problem involving mixed type of constraints in which =, ≥ and ≤ are involved and the objective function is to be maximized.

Example 4.6. *Maximize* $Z = 2x_1 + 4x_2 – 3x_3$

Subject to the constraints

$$x_1 + x_2 + x_3 \geq 8$$
$$x_1 – x_2 \geq 1$$
$$3x_1 + 4x_2 + x_3 \leq 40$$

Solution. The problem can be formulated in the standard form.

Maximize $Z = 2x_1 + 4x_2 – 3x_3 + 0S_1 + 0S_2 – MA_1 – MA_2$

Subject to constraints

$$x_1 + x_2 + x_3 + A_1 = 8$$
$$x_1 – x_2 – S_1 + A_2 = 1$$
$$3x_1 + 4x_2 + x_3 + S_2 = 40$$
$$x_1 \geq 0, x_2 \geq 0, S_1 \geq 0, S_2 \geq 0, A_1 \geq 0, A_2 \geq 0$$

TABLE 4.19 First Simplex Table

		C_j	2	4	–3	0	0	–M	–M	
C_B	Solution mix Variables (B)	Solution values $b (= x_B)$	x_1	x_2	x_3	S_1	S_2	A_1	A_2	Minimum ratio
– M	A_1	8	1	1	1	0	0	1	0	8
– M	A_2	1	(1)	–1	0	–1	0	0	1	1 → key row
0	C_2	40	3	4	1	0	1	0	0	$\frac{40}{3}$
Z_j			– 2M	0	– M	M	0	– M	–M	
$(C_j – Z_j)$			2 + 2M	4	–3 + M	– M	0	0	0	

↑ key column (under x_1)

where A_1 and A_2 are the artificial constraints, S_1 is the surplus variable, S_2 is the slack variable and M is a very large quantity.

For initial basic solution

$$A_1 = 8, A_2 = 1, S_2 = 40$$

This is a problem of maximization, hence we select 2 + 2M, the largest positive number in $(C_j - Z_j)$ x_1 will enter and A_2 will depart. Use the following row operations:

$$R - 2 \text{ (New)} \rightarrow R - 2 \text{ (old)}$$
$$R - 1 \text{ (New)} \rightarrow R - 1 \text{ (old)} - R_2 \text{ (new)}$$
$$R - 3 \text{ (New)} \rightarrow R - 3 \text{ (old)} - 3\, R_2 \text{ (new)}$$

TABLE 4.20 Second Simplex Table

		C_j	2	4	–3	0	0	–M	–M	
C_B	**Solution mix Variables (B)**	**Solution values $b(= x_B)$**	x_1	x_2	x_3	S_1	S_2	A_1	A_2	**Minimum ratio**
– M	A_1	7	0	(2)	1	1	0		–1	$\frac{7}{2}$
2	x_1	1	1	–1	0	–1	0		1	–1
0	S_2	37	0	7	0	3	1		–3	$\frac{37}{7}$
Z_j			2	–2M – 2	– M	– M – 2	0		–M + 2	
$(C_j - Z_j)$			0	6 + 2M	–3 + M	M + 2	0		– 2	

→ key row

↑ key column

$$R - 2 \text{ (new)} = R - 2 \text{ (old)}$$
$$R - 1 \text{ (new)} = R - 1 \text{ (old)} - R - 2 \text{ (new)}$$
$$R - 3 \text{ (new)} = 40 - 3 \times 1 = 37,\ 3 - 3 \times 1 = 0,\ 4 - 3 \times -1 = 7$$
$$0 - 3 \times 0 = 0,\ 0 - 3 \times -1 = 3,\ 1 - 3 \times 0 = 1,\ 0 - 3 \times 1 = -3$$

Now, x_2 will enter as new variable and A_1 will depart as shown. Third Simplex table can be prepared by using the following row operations :

$$R - 1 \text{ (new)} = R - 1 \text{ (old)}$$
$$R - 2 \text{ (new)} = R - 2 \text{ (old)} + R - 1 \text{ (new)}$$
$$R - 3 \text{ (new)} = R - 3 \text{ (old)} - 7\, R - 1 \text{ (new)}$$
$$R - 1 \text{ (new)} = \frac{7}{2}, 0, 1, \frac{1}{2}, \frac{1}{2} \quad 0$$
$$R - 2 \text{ (new)} = \frac{9}{2}, 1, 0, \frac{1}{2}, \frac{-1}{2} \quad 0$$
$$R - 3 \text{ (new)} = 37 - 7 \times \frac{7}{2} = \frac{25}{2},\ 0 - 7 \times 0 = 0, 7 - 7 \times 1 = 0$$
$$0 - 7 \times \frac{1}{2} = \frac{-7}{2},\ 3 - 7 \times \frac{1}{2} = \frac{-1}{2},\ 1 - 7 \times 0 = 1 = \frac{25}{2}, 0, 0, \frac{-7}{2}\ \frac{-1}{2}, 1$$

TABLE 4.21 Third Simplex Table

NOTES

		C_j	2	8	–3	0	0	–M	–M
C_B	**Solution mix Variables (B)**	**Solution values** $b\ (= x_B)$	x_1	x_2	x_3	S_1	S_2	A_1	A_2
4	x_2	$\frac{7}{2}$	0	1	$\frac{1}{2}$	$\frac{1}{2}$	0		
2	x_1	$\frac{9}{2}$	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	0		
0	S_2	$\frac{25}{2}$	0	0	$-\frac{7}{2}$	$-\frac{1}{2}$	1		
Z_j			2	4	3	1	0		
$(C_j - Z_j)$			0	0	– 6	–1	0		

Since all the entries in $C_j - Z_j$ are either 0 or negative, optimal solution has ben obtained wit

$$x_1 = \frac{9}{2},\ x_3 = \frac{7}{2}\ ,x_3 = 0,\ S_2 = \frac{11}{2} \text{ and } Z = 2x_1 + 4x_2 - 3x_3 + 0S_1 + 0S_2$$

$$= 9 + 14 - 0 + 0 + 0 = \text{Rs. } 23.$$

Two Phase Simplex Method

Example 4.14. *Maximize* $Z = 5x_1 + 3x_2$

Subject to constraints

$$2x_1 + x_2 \le 1$$
$$x_1 + 4x_2 \ge 6$$
$$x_1, x_2 \ge 0.$$

Solution. Phase I. It consists of the following steps:

Step 1. Adding slack variables, the problem becomes

$$2x_1 + x_2 + S_1 = 1$$
$$x_1 + 4x_2 - S_2 = 6$$

Step 2. Putting $x_1 = 0$ and $x_2 = 0$

$$S_1 = 1$$
$$S_2 = -6.$$

This gives the initial basic solution. However, it is not a basic feasible solution since S_2 is negative.

So, we will introduce artificial variable A_1 and the above constraint can be written as

$$2x_1 + x_2 + S_1 = 1 \quad ...(i)$$
$$x_1 + 4x_2 - S_2 + A_1 = 6 \quad ...(ii)$$

Step 3. Substituting $x_1 = x_2 = S_2 = 0$ in the constraint equation we get, $S_1 = 1$ and $A_1 = 6$. as the initial basic solution. This can be put in the form of a Simplex tables follows:

NOTES

TABLE 4.22

$C_j \rightarrow$ ↓	Basic Variables	Solution values	0 x_1	0 x_2	0 S_1	0 S_2	1 A_1	Minimum ratio	
0	S_1	1	2	(1)	1	0	0	1	→ key row
1	A_1	6	1	4	0	–1	1	$\frac{3}{2}$	
	Z_j	6	1	4	0	–1	1		
	$(C_j – Z_j)$		2	–4	0	1	1		
				↑ key column					

As $(C_j - Z_j)$ is negative under some columns, the current basic feasible solution can be improved.

S_1 will be replaced by x_2 as x_2 is the key column, and S_1 is the key row, also 1 is the key element.

New row x_2 (old S_1) will be obtained by dividing all the elements by 1.

New row A_1 can be obtained by using the relationship already known.

$$6 - 4 \times 1 = 2,\ 1 - 4 \times 2 = -7,\ 4 - 4 \times 1 = 0,\ 0 - 4 \times 1 = -4$$
$$-1 - 4 \times 0 = -1,\ 1 - 4 \times 0 = 1$$

New Simplex table can be constructed as follows:

TABLE 4.23

$C_j \rightarrow$ ↓	Basic Variables	Solution values	**0** x_1	**0** x_2	**0** S_1	**0** S_2	**1** A_1
0	x_2	1	2	1	1	0	0
1	A_1	2	–7	0	–4	–1	1
	Z_j	2	–7	0	–4	–1	1
	$(C_j – Z_j)$		7	0	4	1	0

Since all the elements are either positive or zero, an optimal basic solution has been arrived.

However, $A_1 = 2$ which is > 0, the given LPP does not possess any feasible solution and the procedure stops.

4.5 SENSITIVITY ANALYSIS

The solution to LPP is based on a number of deterministic assumptions like the prices are known exactly and are fixed, resources are known with certainty and time needed to manufacture/ assemble/produce a product is fixed. In real life situations, which are dynamic and changing, the effect of variation of these variables must be studied and understood. This process of knowing the impact of variables on the outcome of optimal result is known as sensitivity analysis of linear programming problems. Let us say, for example, that if originally we had assumed the cost per unit to be Rs 10 but it turns out be Rs 11, how will the final profit and solution mix vary. Also, if we start with the assumption of certain fixed resources like man hours or machine hours and as we proceed we realise the availability can be improved, how will this change our optimal solution.

Sensitivity analysis can be used to study the impact of changes in:

(*a*) Addition or deletion of variables initially selected.

(*b*) Change in the cost or price of the product under consideration.

(*c*) Increase or decrease in the resources.

Sensitivity analysis uses the following two approaches :

(*a*) It involves solving the entire problem by trial and error approach and involves very cumbersome calculations.

Every time data of a variable is changed, it becomes another set of the problem and has to be solved independently.

(*b*) The last simplex table may be investigated. This reduces completion and computations considerably.

Limitations of Sensitivity Analysis

Sensitivity analysis does take into account the uncertainty element, yet, it suffers from the following limitations:

(*a*) Only one variable can be taken into account at one time. Hence, the impact of many variables changing cannot be considered simultaneously.

(*b*) It suffers from the linearity limitations as only linear relationship between the variable is considered.

(*c*) The extent of uncertainty cannot be studied.

(*d*) As the result can be judged by individual analysts depending upon their skills and experience, it to that extent subjective in nature.

SUMMARY

- Simplex method is an algebraic procedure in which a series of repetitive operations are used and we progressively approach the optimal solution.
- This method developed by the American mathematician G. B. Dantizg, can be used to solve any problem, which has a solution. The process of reaching the optimal solution through different stages is also called iterative.
- The objective is to minimize rather than maximize, a negative ($C_j - Z_j$) value indicates potential improvement. Therefore, the variable associated with largest negative ($C_j - Z_j$) value would be brought into the solution first. Additional variables are brought into set-up such problems.
- The solution to LPP is based on a number of deterministic assumptions like the prices are known exactly and are fixed, resources are known with certainty and time needed to manufacture/assemble/produce a product is fixed. In real life situations, which are dynamic and changing, the effect of variation of these variables must be studied and understood. This process of knowing the impact of variables on the outcome of optimal result is known as sensitivity analysis of linear programming problems.

NOTES

REVIEW AND DISCUSSION QUESTIONS

1. Explain the following terms :

(*a*) Basic feasible solution.

(*b*) Optimal solution.

2. Explain step by step the method used in solving LPP using simplex method.

3. Explain the use of slack, surplus and artificial variables when are these used and why.

4. How are the key column, key row and key element (number) selected ?

5. Explain the use of simplex method in solving the maximization and minimization problems. What are the differences in the approach ?

6. What do you understand by a redundant constraint ? Do these constraints influence analysis and final solution of a LPP ?

7. What are the limitations of LPP ? Give examples to support your argument.

8. What do the coefficient in a simplex table represent ? Why is it necessary to compute a new set of coefficients for each table in the analysis ?

9. Explain the terms decision variables, basic variables, entering and departing variables.

10. Write a detailed note on the sensitivity analysis.

11. Maximize $x_1 + 2x_2 + 3x_3 - x_4$

Subject to
$$x_1 + 2x_2 + 3x_3 = 15$$
$$2x_1 + x_2 + 5x_3 = 20$$
$$x_1 + 2x_2 + x_3 + x_4 = 10$$

Using simplex method.

12. Minimize $Z = 8x_1 + 4x_2 + 2x_3$

Subject to $4x_1 + 2x_2 + x_3 \le 8$
$$3x_1 + 2x_3 \le 10$$
$$x_1 + x_2 + x_3 = 4$$
$$x_1, x_2, x_3 \ge 0$$

Using simplex method.

13. Use Simplex Method to Maximize

$$p = 5x - 2y + 3z$$

Subject to

$$2x + 2y - z \ge 2$$
$$3x - 2y \le 3$$
$$y - 3z \le 5$$
$$x, y, z \ge 0$$

14. Solve the following LPP using simple method:

$$Z = 25x_1 + 80x_2$$

Subject to
$$5x_1 + 6x_2 \ge 15$$
$$9x_1 + 9x_2 \ge 27$$
$$x_1 \ge 0, x_2 \ge 0.$$

NOTES

15. The ABC company makes two products P_1 and P_2 with contribution per unit of Rs. 15 and Rs. 11 respectively. Each of the products is made from two raw materials A and B. P_1 and P_2 require theraw material in the following amounts:

Product	**kgs**	
	A	**B**
P_1	4	3
P_2	2	1
Availability in kgs	400	500

Find the optimum product mix for maximum profit.

16. ABC manufacturing company makes three products x_1, x_2 and x_3 with contribution per unit to profit Rs. 2, Rs. 4 and Rs. 3 respectively. Each of the three product passes through three centres as part production process. Time required in each centre to procedure one unit of ach product is as given below :

Product	**Hours per unit**		
	Centre 1	**Centre 2**	**Centre 3**
X_1	3	2	1
X_2	4	1	3
X_3	2	2	2
Time available (hours)	60	40	80

Determine the optimal mix for next week production.

Is the solution uniqe ? If not give two different solutions.

17. Explain the simplex method by carrying out the iteration in the following problem :

Maximize $Z = 5x_1 + 2x_2 + 3x_3 - x_4 + x_5.$

Subject to $x_1 + 2x_2 + 3x_3 + x_4 = 8$

$3x_1 + 4x_2 + x_3 + x_5 = 7$

x_1 to $x_5 \geq 0.$

18. For the following production given in the table, formulate the poblem as linear programming and solve.

Product	**Machine Time (Hours)**			**Profit per product**
	A	**B**	**C**	
P	8	4	2	20
Q	2	3	0	6
R	3	0	1	8
Available Machine hours per week	250	150	50	

19. Use simplex method to solve :

Maximize $Z = 6x_1 + 4x_2$

Subject to $2x_1 + 3x_2 \leq 30$

$3x_1 + 2x_2 \leq 24$

$x_1 + x_2 \geq 3$

$x_1, x_2 \geq 0.$

NOTES

20. Solve the following LP problem by revised simplex method :

Minimize $Z = -3x_1 + x_2 + x_3$

Subject to
$$x_1 - 2x_2 + x_3 \leq 11$$
$$-4x_1 + x_2 + 2x_3 \geq 3$$
$$2x_1 - x_3 = -1$$
$$x_1, x_2, x_3 \geq 0$$

21. The products A, B and C are produced on three machines centres X, Y and Z. Each product involves operations on each of the machine centres. The time required for each operation for unit amount of each product is as follows :

Products	**Machine Centre**		
	X	**Y**	**Z**
A	10	7	2
B	2	3	4
C	1	2	1

(Time in hours)

There are 100, 77 and 80 hours available at machine centres X, Y and Z respectively.

The profit per unit of A, B and C is Rs 12, Rs 3 and Re 1 respectively. Formulate the problem as LPP (Linear Programming Problem) and find the profit maximization product mix. [IGNOU MBA, Dec 2000]

22. A pharmaceutical company has 100 kgs of A, 180 kgs of B, and 120 kgs of C available per month. They can use these materials to make three basic pharmaceutical products, namely 5 – 10 – 5, 5 – 5 – 10 and 2 – 5 – 10 where the numbers in each case represent the percentage by weight of A, B and C respectively in each of the products. The costs o these raw materials are given below :

Ingredient	**Cost per kg (Rs.)**
A	80
B	20
C	50
Inert ingredient	20

Selling prices of these products are Rs. 40.50, Rs. 43 and Rs. 45 per kg respectively. There is a capacity restriction of the company for product 5 – 10 – 5, so as they cannot produce more than 30 kg per month. Determine how much of each of the product they should produce in order to maximize their monthly profit.

23. Solve the following problem using simplex method :

Maximize $Z = 21x_1 + 15x_2$

Subject to
$$-x_1 - 2x_2 \geq -6$$
$$4x_1 + 3x_2 \leq 12$$

where $x_1, x_2 \geq 0$

24. Maximize $Z = 3x_1 + 8x_2$

Subject to
$$x_1 + x_2 = 200$$
$$x_1 \geq 80$$
$$x_2 \leq 60$$

where $x_1, x_2 \leq 0.$

NOTES

25. The owner of fancy goods shop is interested to determine how many advertisements to release in selected three magazines A, B and C. His main purpose is to advertise in such a way that the total exposure to principal buyers of his goods is maximized. Percentage of readers for each magazine is known. Exposure in any particular magazine is the number of advertisements released multiplied by the number of principal buers. The following data are available:

Particulars	**Magazines**		
	A	**B**	**C**
Reader	1.0 lakh	0.6 lakh	0.4 lakh
Principal buyers	20%	15%	8%
Cost per advertisement	Rs. 8000	Rs. 6000	Rs. 5000

The budgeted amount is at the most Rs. 1.0 lakh for the advertisement. The owner has already decided that magazine A should have no more than 15 advertisement and B and C each gets at least 8 advertisement. Formulate the Linear Programming Problem model and solve it.

26. Maximize $Z = -5x_2$

Subject to $x_1 + x_2 \le 1$

$-0.5\,x_1 - 5x_2 \le -10$

$x_1, x_2 \ge 0$

27. Maximize $Z = 3x_1 + 2x_2$

Subject to $2x_1 + x_2 \le 2$

$3x_1 + 4x_2 \ge 12$

$x_1 \ge 0, x_2 \ge 0$

28. Solve the following problem by simplex method

Maximize $Z = 3x + 2y$

Subject to $-x + 2y \le 4$

$3x + 2y \le 14$

$x - y \le 3$

$x, y \ge 0.$

UNIT 5: LINEAR PROGRAMMING-III

(Duality In Linear Programming)

NOTES

Structure

5.1 CONCEPT OF PRIMAL-DUAL RELATIONSHIP OR DUALITY IN LINEAR PROGRAMMING

The original LPP is called the **Primal**. For every LP problem there exists another related unique LP problem involving the same data which also describes the original problem. The original or primal programme can be solved by transposing or reversing the rows and columns of the statement of the problem. Reversing the rows and columns in this way gives us the dual programme. Solution to dual programme problem can be found out in a similar manner as we use for solving the primal problem. Each LP maximising problem has its corresponding dual, a minimising problem. Also, each LP minimising problem has its corresponding dual, a maximising problem. This duality is an extremely important and interesting feature of Linear Programming Problems (LPP). Important facts of this property are :

(*a*) The optimal solution of the dual gives complete information about the optimal solution of the primal and vice-versa.

(*b*) Sometimes converting the LPP into dual and then solving it gives many advantages, for example, if the primal problem contains a large number of constraints in the form of rows and comparatively a lesser number of variables in the form of columns, the solution can be considerably simplified by converting the original problem into dual and then solving it.

(*c*) Duality can provide us economic information useful to the management. Hence, it has certain far reaching consequences of economic nature, since it helps managers in decision-making.

(*d*) It provides us information as to how the optimal solution changes due to the results of the changes in co-efficient and formulation of the problem. This can be used for sensitivity analysis after optimally tests are carried out.

(*e*) Duality indicates that there is a fairly close relationship between LP and Games Theory as it shows each LPP is equivalent to a two-person zero-sum game.

(*f*) Dual of the dual is a primal.

5.2 DUAL PROBLEMS WHEN PRIMAL IS IN THE STANDARD FORM

We have already seen the characteristics of the standard form of LPP, let us recall them once again. These are:

(*a*) All constraints are expressed in the form of equation, only the non-negativity constraint is expressed as > = 0.

(*b*) The right hand side of each constraint equation is non-negative.

(*c*) All the decision variables are non-negative.

(*d*) The objective function Z, is either to be maximised or minimized.

Let us consider a general problem.

The primal problem can be expressed as

Maximize $Z = C_1 X_1 + C_2 X_2 + \ldots\ldots\ldots + C_n X_n$

Subject to $a_{11} X_1 + a_{12} X_2 + \ldots\ldots\ldots + a_1 n Xn < = b_1$

$a_{21} X_1 + a_{22} X_2 + \ldots\ldots\ldots + a_2 n X_n < = b_2$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$a_{m1} X_1 + am_2 X_2 + \ldots\ldots + a_{mn} X_n < = b_m$

$X_1, X_2, \ldots\ldots, X_n = 0$

The dual can be expressed as follows:

Minimize $Z^* = B_1 Y_1 + B_2 Y_2 + \ldots\ldots, + B_m Y_n$

Subject to $a_{11} Y_1 + a_{12} Y_2 + \ldots\ldots, + a_{m1} Y_m > = C_1$

$a_{12} Y_1 + a_{22} Y_2 + \ldots\ldots, + a_{m2} Y_m > = C_2$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$a_1 n Y_1 + a_2 n Y_2 + \ldots\ldots + a_{mn} Y_m > = C_m$

$Y_1, Y_2, \ldots\ldots\ldots, Y_m > = 0$

where $Y_1, Y_2, \ldots\ldots\ldots, Y_m$ are the dual decision variables.

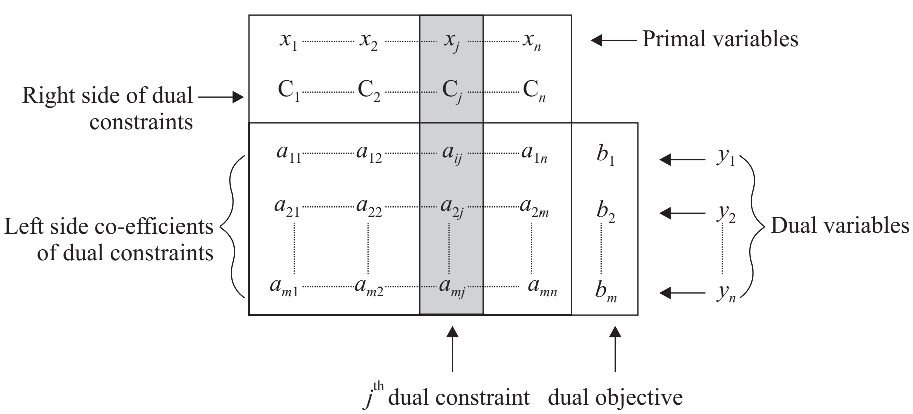
In general, standard form of the primal is defined as

$$\text{Maximize or Minimize } Z = \sum_{j=1}^{n} C_j \, x_j$$

Subject to $\sum_{j=1}^{n} a_{ij} \, x_j = b_i \qquad i = 1, 2, \ldots\ldots\ldots, m$

$X_j > = 0 \qquad j = 1, 2, \ldots\ldots, n$

For constructing a dual of this standard form, let us arrange the co-efficient of prima as



NOTES

It may be noted that dual is obtained symmetrically from the primal using the following rules:

(*a*) For every primal constraint, there is a dual variable, here $X_1, X_2, \ldots\ldots\ldots, X_n$ are the primal constraints and $Y_1, Y_2, \ldots\ldots\ldots, Y_m$ are the dual variables.

(*b*) For every primal variable, there is a dual constraint $X_1, X_2, \ldots\ldots\ldots, X_n$ are the primal variables.

(*c*) The constraint co-efficients of a primal variable form, the left side co-efficients of the corresponding dual constraints, and the objective co-efficient of the same variable becomes the right hand side of the dual constraint as shown above.

The above rules indicate that the dual problem will have *m* variables ($Y_1, Y_2, \ldots\ldots, Y_m$) and *n* constraints (related with $X_1, X_2, \ldots\ldots, X_n$). The sense of optimisation, type of constraints and the sign of dual variables, for the maximisation and minimisation types of standard form are give below.

Standard Primal			Dual		
Objective	**Constraints**	**Variables**	**Objective**	**Constraints**	**Variables**
Maximization	Equations with	All Non-negative	Minimisation	> =	Unrestricted
	Non-negative				
Minimisation	RHS		Maximization	≤ =	Unrestricted

5.3 FORMULATION OF THE DUAL OF THE PRIMAL PROBLEM

The parameters and structure of the primal provides all the information necessary to formulate a dual. The following general observations are useful.

(*a*) The primal is a maximisation problem and the dual is a minimising problem. *The sense of optimisation is always opposite for corresponding primal and dual problems.*

(*b*) The primal consists of two variables and three constraints and dual consists of three variable and two constraints. *The number of variables in the primal always equals the number of constraints in the dual. The number of constraints in the primal always equals the number of variables in the dual.*

(*c*) The objective function co-efficients for x_1 and x_2 in the primal equal the right-hand-side constraints for constraints (1) and (2) in the dual. *The objective function co-efficient for the jth primal variable equals the right-hand-side constraint for the jth dual constraint.*

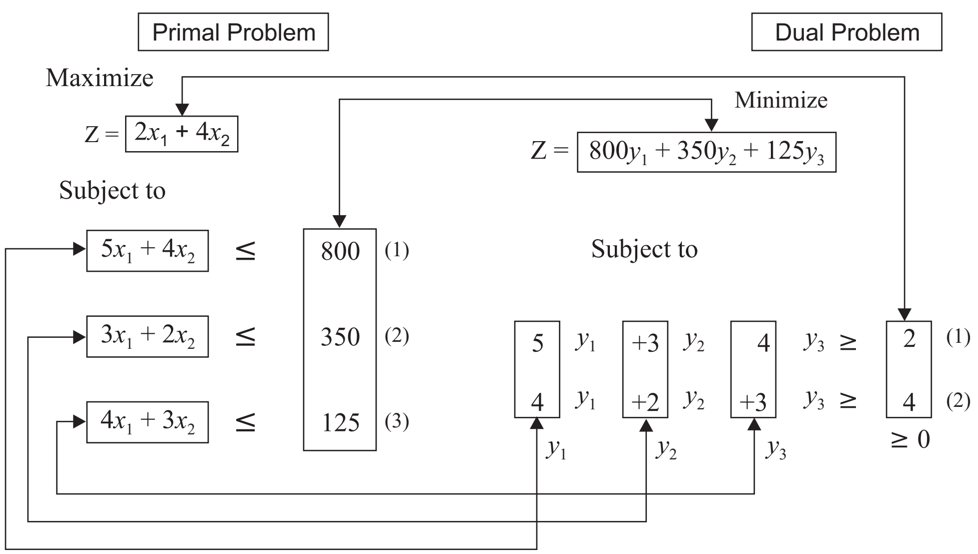
(*d*) The right-hand-side constraints for constraints (1), (2) and (3) in the primal equal the objective function co-efficients for the dual variables y_1, y_2 and y_3. *The right-hand-side constraints for the ith primal constraint equals the objective function coefficient for the ith dual variable.*

(*e*) The variable co-efficients for constraint (1) of the primal equal the column co-efficients for the dual variable y_1. The variable co-efficients of constraints (2) and (3) of the primal equal the column co-efficients of the dual variables y_2 and y_3. *The co-efficients a_{ij} in the primal are transpose of those in the dual. That is, the row co-efficients in the primal become column co-efficients in the dual, and vice-versa.*

The above observations can be summarised in the form of a tablegiven below.

S.No	Maximization Problem		Minimisation Problem
1.	No. of constraints	⇔	No. of variables
2.	(≤) Constraints	⇔	Non-negative variable
3.	(≥) Constraints	⇔	Non-positive variable
4.	(=) Constraints	⇔	Unrestricted variable
5.	Number of variables	⇔	No. of constraints
6.	Non-negative variable	⇔	(≥) Constraints
7.	Non-positive variable	⇔	(≤) Constraints
8.	Unrestricted variable	⇔	(=) Constraints
9.	Objective function co-efficient for *j*th variable	⇔	Right -hand-side constant for *j*th constraint
10.	Right -hand-side constant for *i*th constraint	⇔	Objective function co-efficient for *j*th variable
11.	Co-efficient in constraint *i* for variable	⇔	Co-efficient in constraint *i* for variable *i*

The following figure shows this relationship between primal and dual:



Example 5.1 *The following is a primal problem:*

Minimize $Z = 10\,x_1 + 20\,x_2 + 15\,x_3 + 12\,x_4$

Subject to $x_1 + x_2 + x_3 + x_4 \geq 100$ ... (*i*)

$2x_1 - x_3 + 3\,x_4 \leq 140$ ... (*ii*)

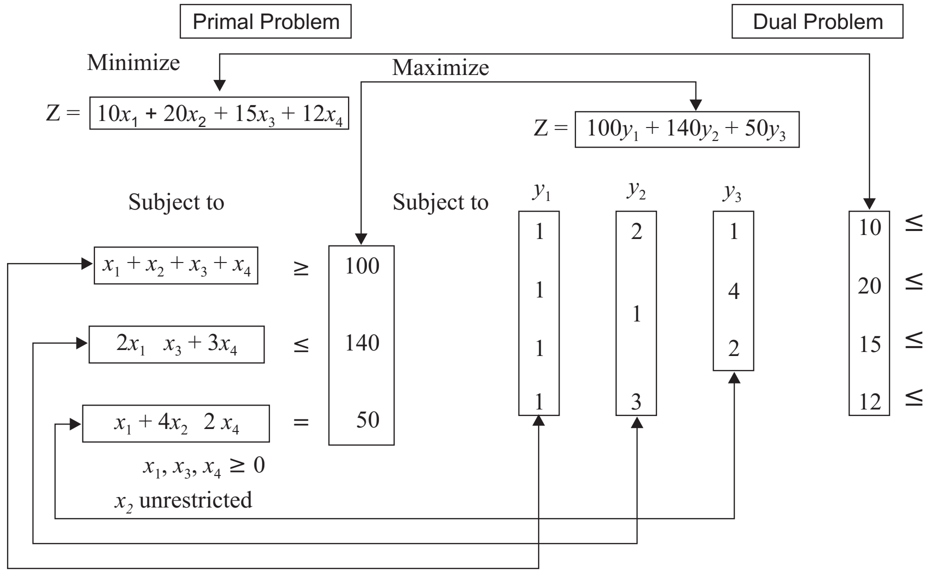
$x_1 + 4x_2 - 2x_4 = 50$ ... (*iii*)

$x_1, x_3, x_4 \geq 0$, x_2 *unrestricted*

Formulate its corresponding duel.

NOTES

Solution.



Dual is

Maximize $\qquad Z = 100\,y_1 + 140\,y_2 + 50\,y_3$

Subject to

$$y_1 + 2y_2 + y_3 \le 10$$
$$y_1 + 4y_3 = 20$$
$$y_1 - y_2 \le 15$$
$$y_1 + 3y_2 - 2y_3 \le 12$$
$$y_1 \ge 0,\ y_2 \le 0,\ y_3 \text{ unrestricted}$$

It has been seen earlier in the table comparing the primal and the dual that an equality constraint in one problem corresponds to an unrestricted variable in the other problem. An unrestricted variable can assume a value which is positive, negative or 0. Similarly, a problem may have non-positive variables. ($x_j \le 0$)

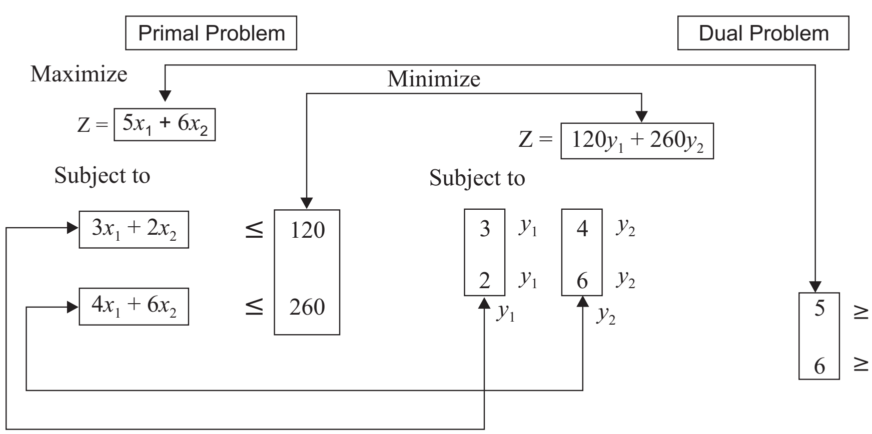
Example 5.4. *Given the following primal problem, formulate the corresponding dual problem.*

Minimize $\qquad Z = 8x_1 + 5x_2 + 6x_3$

Subject to

$$x_1 + x_2 + x_3 = 25$$
$$4x_1 - 5x_2 \ge 10$$
$$x_1 - x_2 + 2x_3 \le 48$$
$$x_2 \le 12$$
$$x_1,\ x_2 \ge 0,\ x_3 - \text{unrestriced}$$

Solution.



Corresponding dual is

Maximize $Z = 25\, y_1 + 10\, y_2 + 48\, y_3$

Subject to $y_1 + 4y_2 + y_3 \geq 8$

$4y_1 - 5y_2 \leq 5$

$y_1 - y_2 + 2y_3 \geq 6$

$y_1, y_2 \geq 0$ y_3 unrestricted

Dual Problem

Let $Y = [y_1, y_2, y_3, y_4]$ be the dual variables, then the dual problem is to determine Y so as to

Minimize $f(Y) = (3, 4, 1, 6)\, [y_1, y_2, y_3, y_4]$

Subject to th contraints

$\geq\ y_1, y_2, y_3, y_4 \geq 0$

or Minimize $f(Y) = 3y_1 + 4y_2 + y_3 + 6y_4$

Subject to the constraints

$$5y_1 - 2y_2 + y_3 - 3y_4 \geq 2$$
$$6y_1 + y_2 - 5y_3 - 3y_4 \geq 5$$
$$-y_1 + 4y_2 + 3y_3 + 7y_4 \geq 6$$

and $y_1, y_2, y_3, y_4 \geq 0$

Example 5.6. *Obtain the dual problem of the following LPP:*

Maximize $Z = x_1 - 2x_2 + 3x_3$

subject to the constraints

$$-2x_1 + x_2 + 3x_3 = 2$$
$$2x_1 + 3x_2 + 4x_3 = 1$$
$$x_1, x_2, x_3 \geq 0$$

Solution. This can be done conveniently by the following method:

Dual is, Minimize $Z = 2y_1 + y_3$ subject to

$$-2y_1 + 2y_2 \geq 1$$
$$-y_1 + 3y_2 \geq -2$$

$3y_1 + 4y_3 \geq 3$ y_1, y_2 are unrestrcted in sign.

As for equal to (=) constraint, the variable is unrestricted.

Example 5.9. *Obtain the dual problem of the following primal problem:*

Minimize $Z = 600\, x_1 + 500\, x_2$

Subject to the constraints

$$3x_1 + x_2 \geq 10$$
$$8x_1 + x_2 \geq 18$$
$$6x_1 + 4x_2 \geq 20$$
$$10x_1 + 20x_2 \geq 30$$
$$x_1, x_2 \geq 0$$

Solution.

Dual problem is

Maximize $Z = 10y_1 + 18y_2 + 20y_3 + 30y_4$

Subject to

$$3y_1 + 8y_2 + 6y_3 + 10y_4 \le 600$$
$$y_1 + y_2 + 4y_3 + 20y_4 \le 500$$
$$y_1, y_2, y_3, y_4 \ge 0$$

NOTES

Example 5.4. *Write the dual of the following LP problem:*

Maximize $Z = 20x_1 + 12x_2 + 16x_3 + 10x_4$

Subject to $3x_1 - 4x_2 + 10x_3 + 6x_4 \le 90$

$$x_1 + x_2 + x_3 = 36$$
$$-2x_2 + 4x_3 + 6x_4 \ge 50$$

and x_4 *unrestricted in sign.*

Solution. When ever an unrestricted variable is provided in primal, it must be converted or expressed as difference of two non-negative variables.

i.e., $x_4 = x_{41} - x_{42}$ where x_{41} and $x_{42} \ge 0$

The given problem can be rewritten as

Maximize $Z = 20x_1 + 12x_2 + 16x_3 + 10\,(x_{41} - x_{42})$

Subject to constraints

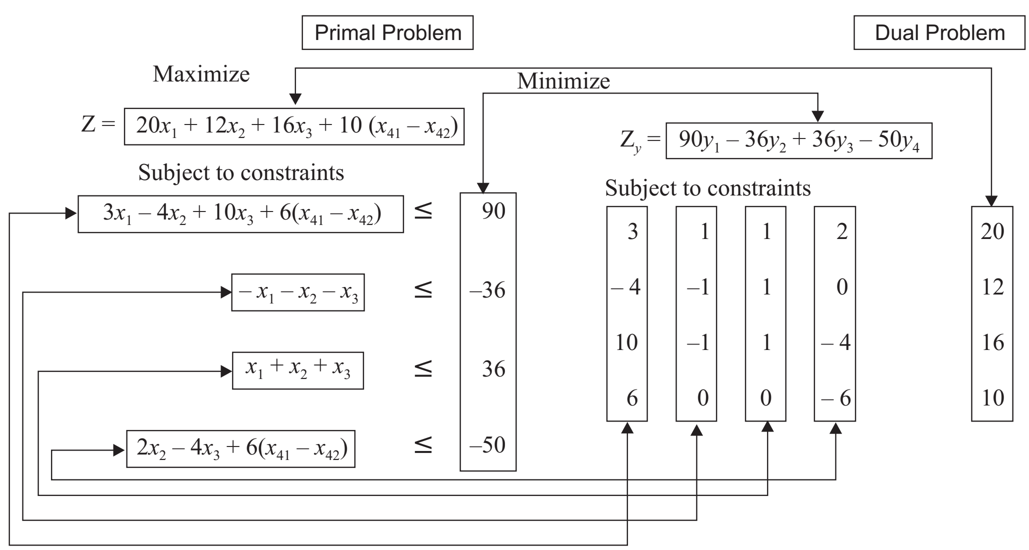
$$3x_1 - 4x_2 + 10x_3 + 6\,(x_{41} - x_{42}) \le 90$$
$$x_1 + x_2 + x_3 \ge 36 \text{ or } -x_1 - x_2 - x_3 \le -36$$
$$x_1 + x_2 + x_3 \le 36$$
$$-2x_2 + 4x_3 + 6\,(x_{41} - x_{42}) \ge 50$$

or $2x_2 - 4x_3 - 6\,(x_{41} - x_{42}) \le -50$

Now, the dual can be formulated as

Dual is $3y_1 + y_2 + y_3 + 2y_4 \ge 20$

$$-4y_1 - y_2 - y_3 \ge 12$$
$$10y_1 - y_2 + y_3 - 4y_4 \ge 16$$
$$6y_1 - 6y_4 \ge 10$$
$$y_1, y_4 \ge 0 y_2, y \text{ unrestricted}$$



5.4 INTERPRETING PRIMAL-DUAL OPTIMAL SOLUTIONS

As has been said earlier, the solution values of the primal can be read directly from the optimal solution table of the dual. The reverse of this also is true. The following two properties of primal-dual should be understood.

Primal-Dual Property 1

If feasible solution exists for both primal and dual the problems, then both the problems have an optimal solution for which the objective function values are equal. A peripheral relationship is that, if one problem has an unbounded solution, its dual has no feasible solution.

Primal-Dual Property 2

The optimal values for decision variables in one problem are read from row (0) of the optimal table for the other problem. The following steps are involved in reading the solution values for the primal from the optimal solution table of the dual:

Step I. The slack-surplus variables in the dual problem are associated with the basic variables of the primal in the optimal solution. Hence, these slack-surplus variables have to be identified in the dual problem.

Step II. Optimal value of basic primal variables can be directly read from the elements in the index row corresponding to the columns of the slack-surplus variables with changed signs.

Step III. Values of the slack variables of the primal can be read from the index row under the non-basic variables of the dual solution with changed signs.

Step IV. Value of the objective function is same for primal and dual problems.

Example 5.5. *Solve the following LPP by using its dual.*

Maximize $Z = 5x_1 - 2x_2 + 3x_3$

Subject to

$$2x_1 + 2x_2 - x_3 \geq 2$$
$$3x_1 - 4x_2 \leq 3$$
$$x_2 + 3x_3 \leq 5$$
$$x_1, x_2, x_3 \geq 0$$

Solution. The problem can be rewritten as

Maximize $Z = 5x_1 - 2x_2 + 3x_3$

Subject to $-2x_1 - 2x_2 + x_3 \leq -2$

(converting ≥ sign into ≤ by multiplying both sides of the equation by – 1)

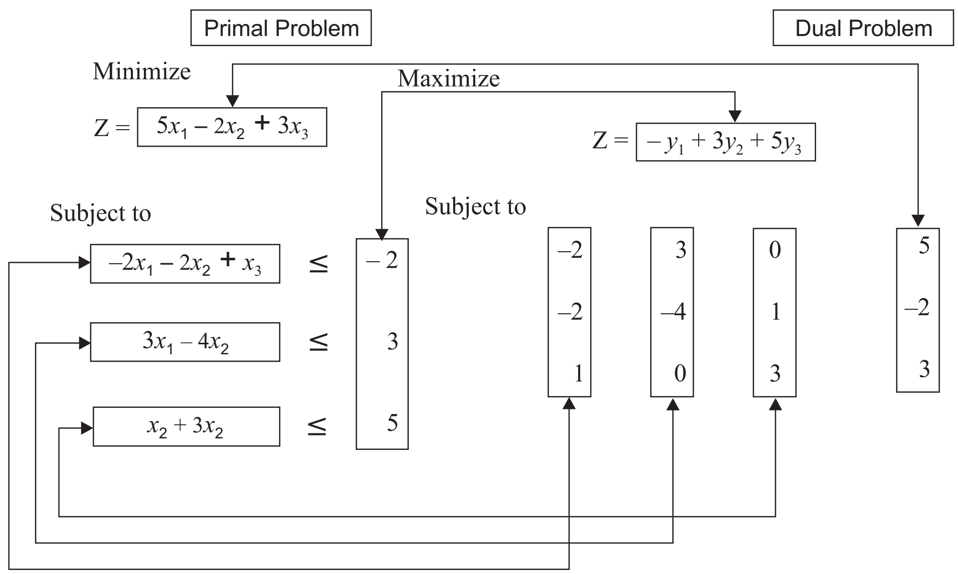
$$3x_1 - 4x_2 \leq 3$$
$$x_2 + 3x_3\ 5$$
$$x_1, x_2, x_3 \geq 0$$

The dual is: Minimize $Z = -2y_1 + 3y_2 + 5y_3$

Subject to the constraints $-2y_1 + 3y_2 \geq 5$

$$-2y_1 - 4y_2 + y_3 \geq -2$$
$$y_1 + 3y_3 \geq 3$$
$$y_1, y_2, y_3 \geq 0$$

NOTES



Step I. Convert the minimisation into a maximisation problem.

Maximize $Z^* = 2y_1 - 3y_2 + 5y_3$

Step II. Make RHS of constraints positive.

$$-2y_1 - 4y_2 + y_3 \geq -2 \text{ is rewritten as}$$
$$2y_1 + 4y_2 - y_3 \leq 2$$

Step III. Make the problem as $N + S$ co-ordinates problem

Maximize $Z^* = 2y_1 - 3y_2 + 5y_3 + 0S_1 + 0S_2 + 0S_3 - MA_1 - MA_3$

Subject to $-2y_1 + 3y_2 - S_1 + A_1 = 5$

$$2y_1 + 4y_2 - y_3 + S_2 = 2$$
$$y_1 + 3y_3 - S_3 + A_3 = 3$$
$$y_1, y_2, y_3, S_1, S_2, S_3, A_1, A_3 \geq 0$$

Step IV. Make N co-ordinates assume 0 values.

Putting $y_1 = y_2 = y_3 = S_1 = S_3 = 0$.

we get $A_1 = 5, S_2 = 2, A_3 = 3$ is the basic feasible solution. This can be represented in the table as follow:

Initial Solution

C_j			2	–3	–5	0	0	0	–M	–M	**Minimum ratio**
C_B	**Basic variable**	**Solution variables**	y_1	y_2	y_3	S_1	S_2	S_3	A_1	A_3	
– M	A_1	5	–2	3	0	–1	0	0	1	0	$\frac{5}{3}$
0	S_2	2	2	(4)	–1	0	1	0	0	0	$\frac{1}{2} \rightarrow$
M	A_3	3	1	0	3	0	0	–1	0	1	∞
Z_J			M	–3M	–3M	M	0	M	–M	–M	
	$(C_j - Z_j)$		2 – M	–3 + 3M	–5 + 3M	–M	0	–M	0	0	

Step V. $C_j - Z_j$ is positive under some columns, it is not the optimal solution. Perform the optimality test.

Step VI. Write second, third or fourth Simplex table unless you come to the optimal solution. This has been provided in the table belo:

Optimal Solution

C_j			2	–3	–5	0	0	0	– M	– M
C_B	**Basic variable**	**Solution variables**	y_1	y_2	y_3	S_1	S_2	S_3	A_1	A_3
0	S_3	11	–15	0	0	– 4	– 3	1	4	– 1
– 3	y_2	$\frac{5}{3}$	$-\frac{2}{3}$	1	0	$-\frac{1}{3}$	0	0	$\frac{1}{3}$	0
– 5	y_3	$\frac{14}{3}$	$-\frac{14}{3}$	0	1	$-\frac{4}{3}$	– 1	0	$\frac{4}{3}$	0
Z_J			$\frac{76}{3}$	–3	–5	$\frac{23}{3}$	5	0	$-\frac{23}{3}$	0
	$(C_j - Z_j)$		$-\frac{70}{3}$	0	0	$-\frac{23}{3}$	– 5	0	$-\frac{23}{3} - M$	–M

Since all the values in $(C_j - Z_j)$ are negative, this is the optimal solution.

$$y = 0, y_2 = \frac{5}{3}, y_3 = \frac{14}{3}$$

$$Z^*_{min} = -Z_{max} = \left(\frac{-85}{3}\right) = \frac{85}{3}$$

5.5 DUAL SIMPLEX METHOD

The basic difference between the regular Simplex Method and the Dual Simplex Method is that whereas the regular Simplex Method starts with basic feasible solution, which is not optimal and it works towards optimality, the dual Simplex Method starts with an infeasible solution which is optimal and works towards feasibility. The following steps are involved in this method:

Step I. Convert the problem into a maximization problem, if initially it is a minimization problem.

Step II. If there are any ≥ type constraints, these must be converted into ≤ type constraints by multiplying both sides by – 1.

Step III. Obtain the initial basic solution – For this, the inequality constraints have to be converted into equality by adding slack variables. Make a dual Simplex table by putting this information in the form of a table.

Step IV. Compute $C_j - Z_j$ for each column.

(*a*) If all $C_j - Z_j$ are negative or zero and all solution values are non-negative, the solution found above is the optimum basic feasible solution.

(*b*) If all $C_j - Z_j$ are negative and zero and at least one value of 'solution value' is negative then proceed to next step, *i.e.*, step V.

(*c*) If any $C_j - Z_j$ is positive, this method cannot be applied.

Step V. Select the row that contains the most negative 'solution value'. This row is called the **key row** or the **pivot row**. The corresponding basic variable leaves the current solution.

Step VI. Scrutinise the elements of the key row.

(*a*) If all elements are non-negative, the problem does not have a feasible solution.

(*b*) If at least one element is negative, find the ratios of corresponding elements of $C_j - Z_j$ row to these elements, ignoring the ratios associated with positive or zero elements of the key row. Select the smallest of these rows. The corresponding column is the key column and the associated variable is the entering variable. Mark (circle) the key element or pivot element.

NOTES

Step VII. Make the key element unity (1). Carryout the row operations as is done in the regular Simplex Method and repeat until.

(*a*) An optimal feasible solution is obtained in a finite number of steps or,

(*b*) An indication of non-existence of feasible solution is found.

Example 5.6. *Use dual simplex method to:*

Maximize $Z = -3x_1 - 2x_2$

Subject to

$$x_1 + x_2 \geq 1$$
$$x_1 + x_2 \leq 7$$
$$x_1 + 2x_2 \geq 10$$
$$x_2 \leq 3 \quad x_1, x_2 \geq 0$$

Solution. The given problem may be put in the form as

Maximize $Z = -3x_1 - 2x_2$

Subject to

$$-x_1 - x_2 \leq -1$$
$$x_1 + x_2 \leq 7$$
$$-x_1 - 2x_2 \leq -10$$
$$x_2 \leq 3$$
$$x_1, x_2 \geq 0$$

Adding slack variables S_1, S_2, S_3 and S_4.

Maximize $Z = -3x_1 - 2x_2 + 0S_1 + 0S_2 + 0S_3 + 0S_4$

Subject to

$$-x_1 - x_2 + S_1 = -1$$
$$x_1 + x_2 + S_2 = 7$$
$$-x_1 - 2x_2 + S_3 = -10$$
$$x_2 + S_4 = 3$$
$$x_1, x_2, S_1, S_2, S_3 \text{ and } S_4 \geq 0$$

This can be put in the form of First Simplex table.

Putting $x_1 = x_2 = 0$, $S_1 = -1$, S= , $S_3 = -10$ and $S_4 = 3$.

$C_j \downarrow$	$C_j \rightarrow$		−3	−2	0	0	0	0	
	Basic variable	**Solution value**	x_1	x_2	S_1	S_2	S_3	S_4	
	S_1	−1	−1	−1	1	0	0	0	
	S_2	7	1	1	0	1	0	0	
	S_3	−10	−1	(−2)	0	0	1	0	→ key row
	S_4	3	0	1	0	0	0	1	
	Z_j	0	0	0	0	0	0	0	
	$(C_j - Z_j)$		−3	−2	0	0	0	0	
				↑ key column					

Selecting the row with most negative solution value, *i.e.*, – 10, this is the key row. To find the key column.

$$\text{For } x \text{ column} = \frac{C_j - Z_j}{\text{element corresponding to } x_1 \text{ in the key row}} = 3$$

$$x_2 \text{ column} = \frac{-2}{-2}$$

Selecting the smaller of these ratios, i.e., x_2 is the key column and – 2 is the key element and – 2 is the key element (circled). Row S_3 is to be replaced by x_2 . Value of x_2 elements is obtained by dividing the row by key element, i.e., – 2. Hence the row x_2 (earlier S_3) is 5, $\frac{1}{2}$, 1, 0, 0, $\frac{-1}{2}$, 0.

New row values of S_1, S_2 and S_4 can be determined by using the relationship.

New row element = (Element in old row) – [(Intersectional element in old row) × (Corresponding element in replacing row)

S_1 row Element solution value = – 1 – (– 1 × 5) = – 1 5 = 4

$$x_1 \text{ element} = -1 - \left(-1 \times \frac{1}{2}\right) = -\frac{1}{2}$$

$$x_2 \text{ element} = -1 - (-1 \times 1) = 0$$

$$S_1 \text{ element} = 1 - (-1 \times 0) = 1$$

$$S_2 \text{ element} = 0 - (-1 \times 0) = 0$$

$$S_3 \text{ element} = 0 - \left(-1 \times \frac{-1}{2}\right) = \frac{1}{2}$$

$$S_4 \text{ element} = 0 - (-1 \times 0) = 0$$

So,new S_1 row elements are: 4, $-\frac{1}{2}$, 0, 1, 0, –, 0.

Similarly, elements of new S_2 row can be found out

New S_2 row elements

Solution value element = 7 – (1 × 5) = 2

$$x_1 \text{ element} = 1 - \left(1 \times \frac{1}{2}\right)$$

$$x_2 \text{ element} = 1 - (1 \times 1) = 0$$

$$S_1 \text{ element} = 0 - (1 \times 0) = 0$$

$$S_2 \text{ element} = 1 - (1 \times 0) = 1$$

$$S_3 \text{ element} = \left(1 \times -\frac{1}{2}\right) 0 -$$

$$S_4 \text{ element} = 0 - (1 \times 0) = 0.$$

New S_4 elements

Solution value element = 3 – (1 × 5) = – 2

$$x_1 \text{ element} = 0 - \left(1 \times \frac{1}{2}\right) = \frac{1}{2}$$

$$x_2 \text{ element} = 1 - (1 \times 1) = 0$$

NOTES

S_1 element = 0 – (1 × 0) = 0

S_2 element = 0 – (1 × 0) = 0

S_3 element = $0 - \left(1 \times \frac{-1}{2}\right) = \frac{1}{2}$

S_4 element = 1 – (1 × 0) = 1

These values can be written in the Second Basic Infesible Solution as follows:

C_j ↓	C_j→		–3	–2	0	0	0	0	
	Basic variable	**Solution variable**	x_1	x_2	S_1	S_2	S_3	S_4	
0	S_1	4	$\frac{1}{2}$	0	1	0	$-\frac{1}{2}$	0	
0	S_2	2	$\frac{1}{2}$	0	0	1	$\frac{1}{2}$	0	
–2	X_2	5	$\frac{1}{2}$	1	0	0	$-\frac{1}{2}$	0	
0	S_4	–2	($\frac{1}{2}$)	0	0	0	$\frac{1}{2}$		→ key row
	Z_j	–10	–1	–2	0	0	1	0	
	$(C_j - Z_j)$		–2	0	0	0	–1	0	
	$\frac{C_j - Z_j}{\text{Corresponding element in Key row}}$		$\frac{2}{\frac{1}{2}} = 4$	0	0	0	–2	0	
			↑ key column						

Key row is marked with the arrow on the right side →

Selecting he smaller of ratio value = $\frac{C_j - Z_j}{\text{Corresponding element in Key row}}$

Key column is marked ↑ in the table.

Key element $-\frac{1}{2}$ is marked with a circle $\left(-\frac{1}{2}\right)$ in the table.

Row S_4 is to be replaced by x_1. All elements of S_4 are divided by the key element out these are the elements of new x_1 row, i.e., 4, 1, 0, 0, 0, – 1, – 2.

New values of S_1 ow

$$\text{Solution variable} = 4\left(-\frac{1}{2} \times 4\right) = 6$$

$$\text{Element } x_1 = \frac{-1}{2} - \left(-\frac{1}{2} \times 1\right) = 0$$

$$x_2 = 0 - \left(-\frac{1}{2} \times 0\right) = 0$$

$$S_1 = 1 - \left(-\frac{1}{2} \times 0\right) = 1$$

$$S_2 = 0 - \left(-\frac{1}{2} \times 0\right) = 0$$

$$S_3 = -\frac{1}{2} - \left(-\frac{1}{2} \times -1\right) = -1$$

$$S_4 = 0 - \left(-\frac{1}{2} \times -2\right) = -1$$

The values are: 6, 0, 0, 1, 0, – 1, – 1.

New values of S_2 row

$$\text{Solution variable} = 2 - \left(\frac{1}{2} \times 4\right) = 0$$

$$\text{Element } x_1 = \frac{1}{2} - \left(\frac{1}{2} \times 1\right) = 0$$

$$x_2 = 0 - \left(\frac{1}{2} \times 0\right) = 0$$

$$S_1 = 0 - \left(\frac{1}{2} \times 0\right) = 0$$

$$S_2 = 1 - \left(\frac{1}{2} \times 0\right) = 1$$

$$S_3 = \frac{1}{2} - \left(\frac{1}{2} \times -1\right) = 1$$

$$S_4 = 0 - \left(\frac{1}{2} \times -2\right) = 1$$

So, the new values are: 0, 0, 0, 0, 1, 1, 1.

New values of x_2 row are:

$$\text{Solution variable} = 5 - \left(\frac{1}{2} \times 4\right) = 3$$

$$\text{Element } x_1 = \frac{1}{2} - \left(\frac{1}{2} \times 1\right) = 0$$

$$x_2 = 1 - \left(\frac{1}{2} \times 0\right) = 1$$

$$S_1 = 0 - \left(\frac{1}{2} \times 0\right) = 0$$

$$S_2 = 0 - \left(\frac{1}{2} \times 0\right) = 0$$

$$S_3 = -\frac{1}{2} - \left(\frac{1}{2} \times -1\right) = 0$$

$$S_4 = 0 - \left(\frac{1}{2} \times -2\right) = 1$$

The new values are 3, 0, 1, 0, 0, 0, 1

These values can be placed in the table to determine whether it turns out to bea feasible solution or not.

NOTES

C_j ↓	$C_j \rightarrow$		–3	–2	0	0	0	0
	Basic variable	**Solution variable**	x_1	x_2	S_1	S_2	S_3	S_4
0	S_1	6	0	0	1	0	1	–1
0	S_2	0	0	0	0	1	1	1
–2	x_2	3	0	1	0	0	0	1
–3	x_1	4	1	0	0	0	–1	–2
	Z_j	–18	–3	–2	0	0	3	4
	$(C_j – Z_j)$		0	0	0	0	–3	–4

Since all the $C_j – Z_j$ values are either zero or negative, this is the optimal feasible solution.

$$x_1 = 4, x_2 = 3$$

$$Z_{max} = -3 \times 4 - 2 \times 3 = -18$$

5.6 SUMMARY

- The original LPP is called the **Primal**. For every LP problem there exists another related unique LP problem involving the same data which also describes the original problem. The original or primal programme can be solved by transposing or reversing the rows and columns of the statement of the problem.
- Also, each LP minimising problem has its corresponding dual, a maximising problem. This duality is an extremely important and interesting feature of Linear Programming Problems (LPP).
- The primal is a maximisation problem and the dual is a minimising problem. *The sense of optimisation is always opposite for corresponding primal and dual problems.*
- The primal consists of two variables and three constraints and dual consists of three variable and two constraints. *The number of variables in the primal always equals the number of constraints in the dual. The number of constraints in the primal always equals the number of variables in the dual.*
- If feasible solution exists for both primal and dual the problems, then both the problems have an optimal solution for which the objective function values are equal.
- The optimal values for decision variables in one problem are read from row (0) of the optimal table for the other problem.
- The basic difference between the regular Simplex Method and the Dual Simplex Method is that whereas the regular Simplex Method starts with basic feasible solution, which is not optimal and it works towards optimality, the dual Simplex Method starts with an infeasible solution which is optimal and works towards feasibility.

5.7 VIEW AND DISCUSSION QUESTIONS

1. Discuss in brief ‘Duality’ in linear programming.
2. Explain the primal-dual relationships.

NOTES

3. State and explain dual L.P.P.
4. Prove that the dual of the dual of a given primal is again primal.
5. State the fundamental properties of duality and prove any one of them.
6. If the *k*th constraint of the primal problem is an equality, then prove that the dual variable wk is unrestricted in sign.
7. If any variable of the primal problems is unrestricted in sign, the corresponding constraint in the dual will be a strict equality. Prove it.
8. State the dual theorem and explain its implications.
9. If either the primal or the dual problem has a finite optimal solution, then the other problem also has finite optimal solution and the values of the two objective functions are equal. Prove this.
10. Prove that the necessary and sufficient condition for any linear programming problem and its dual to have an optimal solution is that both have feasible solutions.
11. Prove that if the primal has an unbounded solution, the dual has either no solution or an unbounded solution.
12. What is the essential difference between regular simplex method and dual simplex method?
13. Develop the theory of dual simplex algorithm.
14. Write a short note on 'Complementary Slackness'.
15. State and prove the complementary slackness theorem for the symmetrical dual problem.
16. What do you understand by 'duality' in linear programming? State and prove the theorem of dual primal relationships.
17. Use dual simplex method to solve the L.P.P:
    Minimize $Z = 2x_1 + 3x_3$ subject to the constraints:
    $2x_1 - x_2 - x_3 \geq 3, x_1 - x_2 + x_3 \geq 2, x_1, x_2, x_3 \geq 0.$
18. Solve the following linear programming problem by dual simplex method:
    Minimize $Z = 2x_1 + 9x_2 + 24x_3 + 8x_4 + 5x_5$ subject to the constraints:
    $x_1 + x_2 + 2x_3 - x_5 - x_6 = 1, -2x_1 + x_3 + x_4 + x_5 - x_7 = 2$
    $x_j \geq 0; j = 1, 2, ..., 7$
19. Use dual simplex method to solve the L.P.P:
    Minimize $Z = x_1 + 2x_2 + 3x_3$ subject to the constraints:
    $x_1 - + x_3 \geq 4, x_1 + x_2 + 2x_3 \leq 8.$
20. Apply the principle of duality to solve the LPP :
    Maximize: $Z = 3x_1 - 2x_2$
    Subject to
    $$x_1 + x_2 \leq 5$$
    $$x_1 \leq 4$$
    $$1 \leq x_2 \leq 6$$
    $$x_1 \geq 0, x_2 \leq 0.$$
21. Minimize $Z = 25X_1 + 10X_2$
    $$X_1 + X_2 = 50$$
    $$X_1 \geq 20$$
    $$X_2 \geq 40$$
    $$X_1, X_2 \geq 0$$
    Write dual and solve.

UNIT 6: THE TRANSPORTATION PROBLEMS

NOTES

Structure

6.1 INTRODUCTION

The transportation model seeks the determination of a transporation plan of a single commodity from a number of sources to a number of destinations. The model must have the following information:

(a) Amount of demand at each destination

(b) Availability at each source

(c) The unit transportation cost of commodity from each source to each destination.

Since we are concerned with only one commodity, the destination can get the commodity from any of the sources. The objective of the problems is to find out the amount (quantity) to be transported from each of the sources to each destination so that the total transportation cost is minimum.

Let us put the definition in mathematical terms.

Let m = number of sources/origin (say plant location)

n = number of destinations

a_i = number of units of the commodity available at source (i = 1, 2, 3....., m)

b_j = number of units required at destination (j = 1, 2, 3, ..., n)

C_{ij} = unit transportation cost for transporting the commodity from source i to destination j.

It is clear that the objective in this type of situation is to determine the number of units to be transported from source i (ai) to destination j(bj) so that the total transportation cost is minimized.

Let X_{ij} = the number of units to be transported from source i to destination j.

Now, we have to find X_{ij} (non-negative value) which satisfies the following constrains :

n = number of destinations

$$\sum_{i=1}^{n} X_{ij} = a_i \text{ for } i = 1, 2, 3, ..., m \text{ (availability at source } i \text{ constraint)}$$

$$\sum_{i=1}^{m} X_{ij} = b_j \text{ for } j = 1, 2, 3, ..., n \text{ (requirement at destination } j \text{ constraint)}$$

The total cost of transportation Z (the objective function) can be written as

$$Z = \sum_{i=1}^{m} \sum_{i=1}^{m} x_{ij} c_{ij}$$

The above objective function is to be minimized with the constraints given above.

It may be seen that the equation representing the constraints as well as that of objective function are linear equations in x_{ij}. Hence, essentially it is a Linear Programming Problem (LPP).

The model makes an assumption to simplify problem, the transportation cost on a given route is directly proportional to the number of units transported. It must be noted that the constraint of availability of commodity at source must specify that all the transportation from the source cannot exceed the supply. Hence,

$$\sum_{i=1}^{n} x_{ij} \le a_i = 1, 2, 3, ..., m$$

Similarly, requirement of the commodity at destination must be equal to or more than the demand, *ie.,*

$$\sum_{i=1}^{m} x_{ij} \le a_i = 1, 2, 3, ..., n$$

In real life situations, supply may not equal demand or exceed it. In such situations, the transportation model needs to be balanced.

6.2 TERMINOLOGY USED IN TRANSPORTATION MODEL

Following are some important terms used in Transportation model:

1. *Feasible solution*: Non-negative values of x_{ij} where i = 1, 2, ..., m and j = 1, 2, ..., n which satisfy the constraints of availability (supply) and requirement (demand) is called the feasible solution to the transportation problem.
2. *Basic feasible solution:* It is the feasible solution that contains only $m + n - 1$ non-negative allocation.
3. *Optimal solution:* A feasible solution is said to be optimal solution when the transportation cost is minimum.
4. *Balanced transportation problem:* A transportation problem in which the total supply from all the sources equals the total demand in all the destinations.

Mathematically, $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$

5. *Unbalanced transportation problem*: Such problems which are not balanced are called unbalanced.

   Mathematically, $\sum_{i-1}^{m} a_i \neq \sum_{j-1}^{n} b_j$

NOTES

6. *Matrix terminology:* In the matrix used in transportation problem, the squares are called *cells*. These cells form 'columns' vertically and 'rows' horizontally. Unit costs are written i the cells.

Warehouses

		1	2	3	4
Plant	A	4	2	10	3
	B	6	8	7	5
Demand		15	7	8	12

The cell located at intersection of row B and column 4 is the one in which the unit cost 5 is written as (B, 4).

6.3 ASSUMPTIONS OF TRANSPORTATION MODEL

Transportation model makes the following basic assumptions :

(*a*) *Availability of commodity:* The supply available at different sources is equal to or more than the total demand of different destinations when it is equal it is called a *balanced problem*.

i.e., $\sum_{i=1}^{m} a_i = \sum_{i=1}^{n} b_j$

(*b*) *Transportation of commodity/items:* The model assumes that all items can be conveniently transported from sources to destinations.

(*c*) *Certainty of per unit transportation cost:* There is a definite cost of transportation of items from sources to destinations.

(*d*) *Independent cost per unit:* The per unit cost is independent of the quantity transported from sources to destinations.

(*e*) *Transportation cost:* Transportation cost on any given route is proportional to the number of units transported.

(*f*) *Objective function:* The objective is to minimize the total transportation cost for the entire organization.

Example 6.1. *A car manufacturing company has plants at cities A, B and C. Its destination centres are located at cities X and Y. The capacity of three plants during the next quarter is 1000, 1500 and 1200 cars. The quarterly demand of the two destination centres is 2300 and 1400 cars. The train transportation cost per car per km is Rs. 2. The chart below shows the distance in km between the plants and the distribution centres.*

X	*Y*
1000	*2690*
1250	*1350*
1275	*850*

How many cars should be transported from which plant to which destination centre to minimize cost ?

Solution. Step I. In the above problem the market distances and the cost of transportation per km is given. This must be convert into the costs.

NOTES

	X	Y
A	2000	5380
B	2500	2700
C	2550	1700

Let X_{ij} be the number of cars transported from plants to destination centres ($x_{ij} \geq 0$) since the total supply (1000 + 1500 + 1200 = 3700) happens to equal the total demand (2300 + 1400 = 3700), the transportation model is balanced.

		X	Y	Supply
	A	X_{11}	X_{12}	1000
Source:	B	X_{11}	X_{12}	1500
	C	X_{11}	X_{12}	1200
	Demand	2300	1400	3700

Step II. Objective is to minimize the cost of transportation.

i.e., $$Z = 2000X_{11} + 5380X_{12} + 2500X_{21} + 2700X_{22} + 2550X_{31} + 1700X_{32}$$

In general, if C_{ij} is the unit cost of transportation from *i*th source to *j*th destination, the objective is:

$$\text{Minimize} \sum_{i=1}^{m} \sum_{i=1}^{n} C_{ij} X_{ij}$$

Step III. Constraints

(a) Availability or supply of cars

$$X_{11} + X_{12} = 1000 \text{ (Source A)}$$
$$X_{21} + X_{22} = 1500 \text{ (Source B)}$$
$$X_{31} + X_{32} = 1200 \text{ (Source C)}$$

It may be seen that the constraints are equal to number of plants, *i.e.,* there are three constraints.

(*b*) Requirement of demand of cars at destination centres

$$X_{11} + X_{21} + X_{31} = 2300 \text{ (Destination centre X)}$$
$$X_{12} + X_{22} + X_{32} = 1400 \text{ (Destination centre Y)}$$

It may be seen that in this problem there are ($3 \times 2 = 6$) variables and ($3 + 2 = 5$) constraints. In general, such a solution will involve ($m \times n$) variables and ($m + n$) constraints.

It is also clear that the objective function and the constraint equations are linear in nature, hence this problem can be solved by simplex method of linear programming. However, since 6 variables are involved (in real life situations there will be much more) the calculations will be very long and time-consuming requiring the help of computers. Also, the simplex method is more suitable for maximization problems whereas the transportation problem require the minimization of the objective function.

6.4 SOLUTION OF THE TRANSPORTATION MODEL

Step I. *Make a transportation model*

NOTES

Distribution centres

		X	Y	Supply
	A	Rs. 2000 $X_{11} = 1000$	$X_{11} = 5380$	1000
Plants	B	Rs. 2500 $X_{21} = 1300$	Rs. 2700 $X_{21} = 1400$	+ 1500
	C	Rs. 2550	Rs. 2550	+ 1200
	↑ Demand	2300 +	1400	= 3700

The above problem is balanced or self-contained wherever it is not so, a dummy source or destination, as the case may be, is created to balance the supply and demand.

Step II. *Finding a basic feasible solution*

Basic feasible solution can be found out by using different methods. One technique developed by Dantzig is called the *'North West Corner Rule'*.

6.5 NORTH-WEST CORNER RULE

In this method, we start with the North-West Corner (top left) and allocate the maximum amount allowable by the supply and demand to this variable, *i.e.*, X_{11}. The satisfied column/row is then crossed, meaning that remaining variables in this column/row equal zero. If a column and a row are satisfied simultaneously, only one of them is crossed out. After adjustment of the quantities of supply and that of demand for all left over (uncrossed-out) rows and columns, the minimum possible column/row is allotted to the first uncrossed out element in the new column/row. The process gets completed when exactly one row or one column is left uncrossed out.

The procedure can be explained specifically in the following steps:

Step I. Start the North-West (top left) corner and compare the supply of source 1 (S_1) with the demand of destination centre 1 (D_1). Three conditions are possible.

(a) $D_1 \leq S_1$ it means that the demand at destination centre D_1 is less than the supply at source S_1. In X_{11} (North-West/top left corner) set X_{11} equal to D_1 and proceed horizontally.

(b) $D_1 = S_1$, *i.e.*, the demand is equal to supply, then set X_{11} equal to D_1 and **proceed diagonally.**

(c) $D_1 > S_1$, *i.e.*, the demand is more than supply, then set $X_{11} = S_1$ and **proceed vertically.**

Step II. Proceed in this manner, step by step till a value is allotted to South-East right bottom corner.

The North-West corner rule can be best demonstrated by the example in hand.

1. Set X_{11} = 1000, i.e., the smaller of the amount available at S_1(1000) and that needed at D_1 (2300).
2. Proceed to cell (BX) as per rule (c) above which demands that you should proceed vertically. If $D_1 > S_1$ compare the quantity available at S_2 (1500) with the amount required. Quantity available at D_1(2300 – 1000 = 1300) and set X_{21} = 1300.

3. Proceed to cell BY (rule above) as now D < S. Here S_2 is 1500 and the demand is 1400. So, set X_{22} = 1400. We are required to proceed horizontally to next cell. Since there is no other horizontal cell, the allocation ends here.

The transportation cost associated with the solution is

$$Z = 2000 \times 1000 + 2500 \times 1300 + 2700 \times 1400.$$
$$= 2000000 + 3250000 + 3780000$$
$$= 9030000.$$

6.6 ROW-MINIMA METHOD

In this method, we allocate maximum possible in the lowest cost cell of the first row. The idea is to exhaust either the capacity of the first source or the demand at destination centre is satisfied or both. Continue the process for the other reduced transportation costs until all the supply and demand conditions are satisfied.

In the above problem, we first allot in cell AX of first row as it has the lowest cost of Rs. 2000. So, we allocate minimum out of (1000, 2200), i.e., 1000. This exhausts the supply capacity of plant A and thus the first row is crossed off. The next allocation is in cell BX as the minimum cost in row 2 is in this cell. We allocate minimum of (1500, 1300), i.e., 1300 in this cell. This exhausts the demand requirements of destination centre X and so column 1 is crossed off.

Distribution centres

		X	Y	Supply
	A	Rs. 2000 1000	Rs. 5380	1000
Plants	B	Rs. 2000 1300	Rs. 2700 200	1500
	C	Rs. 2550	Rs. 1700 1200	1200
Demand		2300	1400	= 3700

Now, we proceed to row No. 3 in which the minimum cost Rs. 1700 is in cell CY. Here we allot minimum out of 1400 and 1200. Since the demand of distribution centres is 1400 and we have allotted only 1200 we allot 200 in cell BY. Now, column Y is satisfied and we cross out column Y. Also since in row two the complete supply of 1500 is satisfied (1300 + 200 = 1500) row two is also satisfied and can be crossed out. Similarly, row three is also satisfied and can be crossed out.

Hence Z =Rs. $(2000 \times 1000 + 2500 \times 1300 + 2700 \times 200 + 1700 \times 1200)$ = Rs. 7830000.

6.7 COLUMN MINIMA METHOD

In this method, we start with the first column and allocate as much as possible in the lowest cost cell of this column, so that either the demand of the first destination centre is satisfied or the capacity of the second plant is exhausted or both. There are three cases :

(a) If the demand of first distribution centre is satisfied, cross off the first column and move to the second column on the right.

NOTES

(b) If the supply (capacity) of the *i*th plant is satisfied, cross off the *i*th row and reconsider, the first column with the remaining demand.

(c) If the demand (requirement) of the first distribution centre as also the capacity of *i*th plant are completely satisfied, make a zero allotment in the second lowest cost cell of the first column. Cross off the column as well as the *i*th row and move to the second column.

Continue the process for the resulting reduced transportation table till all the conditions are satisfied. The matrix below shows the solution with this method which is similar to Row Minima method

Distribution centres

		X	Y	Supply
Plants	A	Rs. 2000 1000	Rs. 5380	1000
	B	Rs. 2500 1300	Rs. 2700 200	1500
	C	Rs. 2550	Rs. 1700 1200	1200
	Demand	2300	1400	3700

Let us solve the given problem with the help of this method. Lowest cost cell in the column is AX. We allocate minimum, *i.e.*, 1000 out of 2300, 1000. With this the capacity of plant A is exhausted and thus row one is crossed off. The next allocation is made in cell BX as it now has the minimum, cost of Rs. 2500 in the first column. We allocate minimum 1300 in this cell. Now, the demand of distribution centre X is satisfied we can cross the first column.

Now, we move to the second column in this minimum cost cell is CY. Allocate 1200 in this cell out of 1400 and 1200. Consider the next least cost cell in this column which is BY in which we can allot only 200. Now all the conditions are satisfied.

Transportation cost associated with this solution is

$$Z = \text{Rs. } (2000 \times 1000 + 2500 \times 1300 + 1700 \times 1200 + 2700 \times 200)$$
$$= \text{Rs. } 7830000$$

which is same as obtained with solution by row-minima method.

6.8 LEAST COST METHOD

In this method, we allocate as much as possible in the lowest cost cell or cells and then move to the next lowest cost cell/cells and so on. Let us solve the above problem using the least cost method.

Distribution centres

		X	Y	Supply
Plants	A	Rs. 2000	Rs. 5380	1000
		$X_{11} = 1000$	$X_{12} = 0$	
	B	Rs. 2500	Rs. 2700	1500
		$X_{21} = 1300$	$X_{22} = 200$	
	C	Rs. 2550	Rs. 1700	1200
		$X_{31} = 0$	$X_{32} = 200$	
	Demand	2300	1400	

Here the lowest cost cell is CY (Rs. 1700) and maximum possible allocation, meeting supply and demand requirement is made here, *i.e.*, 1200. This meets the supply position of row 3 and hence it is crossed out.

The next least cost cell is AX (2000). Maximum possible allocation of 1000 is made here and row one is crossed out. Next lowest cost cell is BX (2500) and maximum possible allocation of 1300 is made here as the total demand in column X is 2300 and we have already allocated 1000 in cell AX. Next lowest cost cell is CX (2550) only 0 can be allocated here to meet the demand (2300) and supply (1200) position. Next cell with lowest cost is BY (2700). Here allocation of 200 is possible. The next lowest cost cell is AY where only 0 allocation is possible.

Hence,

$$Z = \text{Rs. } (2000 \times 1000 + 2500 \times 1300 + 2700 \times 200 + 1700 \times 1200)$$
$$= \text{Rs. } 7830000.$$

6.9 VOGEL'S APPROXIMATION METHOD (VAM)

This method usually provides a better initial (starting) solution than the methods described already. In fact, VAM generally, yields an optimum or very close to optimum starting solution. This method takes into account not only the lest cost C_{ij} but also the costs that just exceeds c_{ij}. The following steps are involved in this method.

Step I. Write down the cos matrix as shown below.

		Distribution centres				Supply
	A	X_{11}	Rs. 2000	X_{12}	Rs. 5380	1000 (3380)
Plants	B	X_{21}	Rs. 2500	X_{22}	Rs. 2700	1500 (200)
	C	X_{23}	Rs. 2550	X_{32}	Rs. 1700	1200 (850)
Demand		2300 (500)			1400 (1000)	

Find out the difference between the smallest and second smallest cost elements in each column and write it below the column in brackets, *i.e.*, in column X the difference is 500 and in the second column it is 1000.

Find out the difference between the smallest and second smallest cost elements in each row and write it on the right side of each row in brackets, *i.e.,* in row A 3380, in row B 200 and in row C 850.

It may be noted that the 'difference,' which is indicated under columns or rows actually indicates the unit penalty incurred by failing to make an allocation to the least cost cell in the row or column.

Step II. Select the row or column with the maximum difference and allocate as much as possible (keeping the restrictions of supply and demand in mind) to the least cost cell in the row or column selected. In case of a tie, take up any one. Now, in this example, since 3380 is the greatest difference, we choose row A and allocate 1000 to least cost cell, *i.e.*, AX.

Step III. Cross out the row or column which satisfies the condition by allocation just made. So, row A is crossed out. The matrix without ow A is as shown below.

NOTES

	X	Y	
B	Rs. 2500 X_{21}	Rs. 2500 X_{22}	1500 (200)
C	Rs. 2550 X_{23}	Rs. 2550 X_{32} = 1200	1200 (850)

Repeat steps I to II till all the allocations have been made. Now, column Y shows maximum difference, so we allocate to the least cost cell in Y column, *i.e.,* CY an amount of 1200 but this does not satisfy column Y completely.

Also, row C shows maximum difference (850) out of the two rows. We allocate 1200 to cell CY which is the least cost cell in row C. Since this allocation completely satisfies row C, we cross row C and the shrunke matrix is shown below.

	X	Y
B	Rs. 2500 X_{21} = 1300	Rs. 2500 X_{22} = 200

Since cell BX has the least cost, maximum possible allocation of 1300 is made here. In cell BY, we allocate 200.

All the above allocations made can now be shown in onesingle matrix as below.

Rs. 2000 X_{11} 1000	Rs. 5380 X_{11}
Rs. 2500 X_{21} 1300	Rs. 2700 X_{22} 200
Rs. 2500 X_{21}	Rs. 1700 X_{32} 1200

The cost of transportation associated with this solution is

$$Z = \text{Rs. } (2000 \times 1000 + 2500 \times 1300 + 1700 \times 1200 + 2700 \times 200) = \text{Rs. } 7830000.$$

6.10 PERFORMING OPTIMALITY TEST

We have found out a feasible solution. Now, we must find out whether this feasible solution is optimal or not. Such an optimality test can be performed only on such feasible solutions where

1. The number of allocation is $m + n - 1$

where m = number of rows and n = number of columns.

In given problem $m = 3$ and $n = 2$ so number of allocations is 4 which is the actual case hence optimality test can be applied. Also, all the allocations are independent of each other.

We can test the optimality of a feasible solution by carrying out an examination of each vacant cell to find out whether or not an allocation in that cell reduces the total transportation cost. This can be done by the use of the following two methods:

The Stepping-Stone Method

Let us consider the matrix of the above problem where we have already found ut the feasible solution.

Distribution centres

		X	Y
	A	X_{11} Rs. 2000 [1000]	X_{12} Rs. 5380
Plants	B	X_{21} Rs. 2500 1300 [–100]	X_{11} Rs. 2700 2000 [+100]
	C	X_{23} Rs. 2550 +100	X_{11} Rs. 1700 1200 [–100]

Let us take up any arbitrary empty cell, *i.e.*, CX and allocate +100 units to this cell. Now in order to maintain the restrictions of column X, we must allocate –100 to cell BX and to maintain the row B restriction we must allocate +100 to cell BY. This will result in unbalance of column Y conditions and so we must allot –100 to cell CY.

Now, let us work out the net change in the transportation cost by the changes we have made in allocations.

$$\text{Evaluation of cell CX} = \text{Rs.}(2550 \times 100 - 2500 \times 100 + 2700 \times 100 - 1700 \times 100)$$
$$= 255000 - 250000 + 270000 - 170000$$
$$= \text{Rs. } 105000$$

As the evaluation of the empty cell CX results in a positive value the total transportation cost cannot be reduced. The feasible solution is an optimal solution already.

We must carryout evaluation of all the empty cells to be sure that optimal solution has been arrived. The total number of empty cells are $m \times n - (m + n - 1) = (m - 1)(n - 1)$. Hence $(m - 1)(n - 1)$ cells must be evaluated. In the present problem $m = 3$ and $n = 2$, so only two empty cells are there but in other problems, the number of empty cells could be much more and this procedure becomes very lengthy and cumbersome.

The Modified Distribution (MODI) Method or UV Method

The problem encountered in the stepping stone method of optimality test can be overcome by MODI method because we don't have to evaluate the empty cells one by one, all of them can be evaluated simultaneously. This is considerably time saving. The method has the following steps:

Step I. Set-up the cost matrix of the problem only with the costs in those cells in which allocations have been made.

	X	**Y**
A	Rs. 2000	
B	Rs. 2500	Rs. 2700
C		Rs. 1700

Step II. Let there be set of number V_j (V_1, V_2) across the top of the matrix and a set of number U; (U_1, U_2, U_3) across the left side so that their sums equal the costs entere in the matrix shown above.

NOTES

		V1 = 0	200 V2 = 200
2000	U1	Rs. 2000	
2500	U2	Rs. 2500	Rs. 2700
1500	U3		Rs. 1700

$U_1 + V_1 = 2000$ $\qquad U_2 + V_2 = 2700$

$U_2 + V_1 = 2500$ $\qquad U_3 + V_2 = 1700$

Let $V_1 = 0$ then $U_1 = 2000$, $\qquad U_2 = 2500$

$V_2 = 2700 - 2500 = 200$

$U_3 = 1500$

Step III. Leave the already filled cells vacant and fill the vacant cells with sums of U_i and $V_{j.}$ This is shown in the matrix below.

		0 V_1	200 V_2
2000	U_1	...	2200 $(V_1 + V_2)$
2500	U_2	...	...
1500	U_3	1500 $(U_3 + V_1)$	...

Step IV. Subtract the vacant values now filled in step III from the original cost matrix. This will result in cell evaluation matrix and is shown beow for the example in hand.

...	5380 – 2200 = 3180
...	...
2550 – 1500 = 1050	...

Step V. If any of the cell evaluation turns out to be negative, then the feasible solution is not optimal. If the values are positive the solution is optimal. In the present example, since both the cell evaluation values are positive, the feasible solution is optimal.

Let us take another example where some of the evolutions turns out to be negative to explain the entire procedure.

Let us assume the following transportaion model for this purpose:

Distribution centres

		P	Q	R	S	Supply
	A	200	300	1100	700	6 (100)
Plants	B	100	0	600	100 (1)	1 (100) 10 (300)
	C	500	800	1500	900	17 (Total)
	Demand	7 (100)	5 (300)	3 (500)	2 (600)	

6.11 FEASIBLE SOLUTION BY VAM

Step I. In column P, the difference between the two lowest cost elements is 100 which is entered as (100) below column P. Similarly, the two smallest elements in row Q are 0 and 300. The difference is 300. We write their difference as (300) below column Q. Under column R, we write (500) and under column S we write (600). Similarly, against row A, we write (100), against row B (100) and against row C (300).

Step II. We choose column S (having largest difference 600). In this column, cell BS has the lowest cost, *i.e.*, 100 and we allot 1 as maximum possible allocation of only 1 is possible.

Step III. Cross out row B as the supply 1 is completely satisfied by the allocation made, 1, in step II.

Step IV. Write down the shrunken matrix after crossing out row B as follows:

	P	Q	R	S	Supply
A	200	300 (5)	1100	700	6 (100)
C	500	800	1500	900	10 (300)
Demand	7	5	3	2	
	(300)	(500)	(400)	200	

We repeat step I and write the difference in rows and columns as shown above. In column Q least cost is AQ, we make allocation of 5. Since this satisfies the condition of column Q completely, we cross out column Q and shrunken atrix is written as follows :

	P	R	S	Supply
A	200 (1)	1100	700	1 (500)
C	500	1500	900	10 (400)
Demand	7	3	1	
	(300)	(400)	(200)	

Once again step I is repeated and the difference in rows and columns are written as shown above. We now make allocations in cell AP as this is the least cost cell. Only 1 can be allotted in this cell since this satisfies row A, it is crossed off and the shrunken matrix is rewritten as follows:

	P	R	S	Supply
C	500	1500	900	
	(6)	(3)	(1)	10 (400)
Demand	6	3	1	

In this, as cell CP has the lowest cost, maximum possible allocation of 6 is made here. Next lowest least cost cell is CS and 1 is allotted here and 3 in the cell CR.

The allocations made above are shown in the allocation matrix given below.

		P	Q	R	S	Supply
	A	200 (1)	300 (5)	1100	700	6
Plant :	B	100	0	600	100 (1)	1
	C	500 (6)	800	1500 (3)	900 (1)	10
	Demand	7	5	3	2	

$Z = \text{Rs. } (200 \times 1 + 300 \times 5 + 100 \times 1 + 500 \times 900 \times 1 = 6 + 1500 \times 3)$
$= \text{Rs. } 10200$

NOTES

6.12 OPTIMALITY TEST BY MODI METHOD OR UV METHOD

Step I. Set-up cost matrix only for cells in which allocation have been made.

$V_j \rightarrow$ ↓ U_i

	P	Q	R	S
A	200	300		
B				100
C	500		1500	900

Step II. Enter a set of numbers V_j across the top of the matrix and a set of numbers U_i across the left side so that their sum is equal to the cost entered in step I.

$U_1 + V_1 = 200$ $U_3 + V_4 = 900$
$U_1 + V_2 = 300$ $U_2 + V_4 = 100$
$U_3 + V_1 = 500$ $U_3 + V_3 = 1500$
If $V_1 = 0$, $U_1 = 200$ $V_2 = 300$, $U_3 = 500$, $V_4 = 400$
$U_2 = -300$, $V_3 = 1000$

So the matrix may be rewritten as

V_j/U_i	$V_1 = 0$	100	1000	400
200	200	300		
–300				100
500	500		1500	900

Step III. Let us fill the vacant cells with the sums of U_i and V_j. This is shown below.

V_j/U_i	0	100	1000	400
200	...	...	1200	600
–300	–300	–200	700	...
500	...	600	...	...

Step IV. Now let us subtract the cell values of the matrix of step from the original cost matrix.

…	…	1100 – 1200	1700 – 600
100 + 300	0 + 200	600 – 700	…
…	800 – 600	…	…

	P	Q	R	S
A	...	...	–100	100
B	400	200	–100	...
C	...	200	...	...

This is called the *cell evaluation matrix.*

Step V. Now since two of the cell evaluations are negative, it means the basic feasible solution is not optimal. Hence, we will take steps to find an optimal solution.

Step VI. Identify in the evaluation cell, the cell with most negative entry. In the present example there are two cells, *i.e.*, AR and BR cells with same negative values of –100. So, let us take cell AR.

Step VII. Write the initial feasible solution in the matrix. The cell value with most negative value is called the *identified cell* and is marked (✓).

Step VIII. Trace a path in this matrix consisting of a series of alternatively horizontal and vertical lines. The path begins and terminates in the identified cell. All corners of the path lie in the cells for which allocation have already been made. As the path has to begin and end at the identified cell, it may skip over any number of occupied or vacant cells. This is shown in the table below.

	P	Q	R	S	Supply
A	–1 (1)	(5)	+1		6
B				(1)	1
C	(6) +1		(3) –1	(1)	10
	7	5	3	2	

Step IX. Mark the identified cell (AR) as positive and each occupied cell at the corners of the path alternatively positive and negative and so on.

Step X. Make a new allocation in the identified cell (AR) by entering the minimum allocation on the path that has been assigned a negative sign. Now, add or subtract as the case may be, these new allocation from the original values of the cells on the corners of the path traced, keeping the row and column requirement at the back of the mind. This makes one basic cell as zero and the other cells become non-negative. That basic cell whose allocation has become zero (AP in this case) leaves the solution. The matrix of the second feasible solution can be rewritten as:

	P	Q	R	S	Supply
A		300 (5)	1200 (1)		6
B				100 (1)	1
C	500 (7)		1500 (2)	900 (1)	10
	7	5	3	2	

The total cost for this feasible solution is

$$= \text{Rs. } (300 \times 5 + 1200 \times 1 + 100 \times 1 + 500 \times 7 + 1500 \times 2 + 900 \times 1)$$
$$= \text{Rs. } 10200$$

which is less than the cost found in the original feasible solution.

NOTES

Example 6.2. *Find the feasible solution of the following transportation probem using North-West corner method.*

	W_1	W_2	W_3	W_4	*Supply*
F_1	*14*	*25*	*45*	*5*	*6*
F_2	*65*	*25*	*35*	*55*	*8*
F_3	*35*	*3*	*65*	*15*	*16*
Demand	*4*	*7*	*6*	*13*	

Solution. Initial feasible solution

	W_1	W_2	W_3	W_4	Supply
F_1	14 (4)	25 (2)	45	5	6
F_2	65	25 (5)	35 (3)	55	8
F_3	35	3	65 (3)	15 (13)	16
Demand	4	7	6	13	30 / 30

Since requirement (4 + 7 + 6 + 13) is equal to the supply (6 + 8 + 16) it is a balanced problem.

Step I. Set F_1W_1 (*i.e.,* North-West corner cell) = 4, the smaller amount, Here $S_1 = 6$ and $D_1 = 4$ and so proceed to cell F_1 W_2 as $D < S$.

Step II. Compare the number of units available in F_1 and row (2) and the demand in column W_2 (7) and so set 2 in row F_1 W_2. Since $D > S$ in this case, we have to proceed vertically, so move to cell F_2W_2.

Step III. Here supply is 8 and demand is 5. So, we set 5 in cell F_2W_2 and proceed horizontally ($D < S$) to cell F_2 W_3.

Step IV. In cell F_2W_3, supply is 3 and demand is 6 so we set 3 in cell F_2W_3 and proceed vertically ($D > S$) to cell F_3W_3.

Step V. In cell F_3W_3 the demand is 16 and requirement is 3 so we set 3 in F_3W_3.

Step VI. Allocate 13 in cell F_3W_4.

North-West Corner Method

F_1W_1 $14 \times 4 = 56$

F_1W_2 $25 \times 2 = 50$

F_2W_2 $25 \times 5 = 125$

F_2 W_3 $35 \times 3 = 105$

F_3 W_3 $65 \times 3 = 195$

F_3 W_4 $15 \times 13 = 195$

Total cost = 726

Example 6.3 *Find the initial basic feasible solution to the following transportation problem by*

(a) Minimum cost method

(b) North-west corner rule.

Stte which of the methods is better.

		To :			
		P	*Q*	*R*	*Supply*
From :	*A*	*2*	*7*	*4*	*5*
	B	*3*	*3*	*1*	*8*
	C	*5*	*4*	*7*	*7*
	D	*1*	*6*	*2*	*14*
	Demand	*7*	*9*	*18*	*34*

Solution. Initial basic feasible solution is shown below.

	X	Y	Z	
A	2	7 (2)	4 (3)	5
B	3	3	1 (8)	8
C	5	4 (7)	7	7
D	1 (7)	6	2 (7)	14
Demand	7	9	18	34

(a) Minimum cost method

The lowest cost cells are BR and DP let us allot 7 in cell DP and 8 in cell BR. Now, we move to cells AP and DR as both have the next lowest cost *i.e.*, 2. In cell AP only 0 can be allotted. In cell DR we can allot 7. The next minimum cost cells are BP and BQ in BP we can allot only 0 similarly in BQ we can allot only 0. The next minimum cost cells are CQ and AR. In cell CQ we can allot 7 and in cell AR we can allot 3.

The next minimum cost cell is CP with cost 5. In this cell we allocate 0. In next lowest cost cell DQ, we can allot 0. The next lowest cost cells are AQ and CR. In AQ we allot 2 and in CR we allot 0.

The cost of transportation associated with this solution is

$$Z = \text{Rs. } (7 \times 2 + 4 \times 3 + 1 \times 8 + 4 \times 7 + 1 \times 7 + 2 \times 7)$$
$$= \text{Rs. } (14 + 12 + 8 + 28 + 7 + 14) = \text{Rs. } 83$$

(*b*) **North-West corner rule**

Step I. We start with cell AP (top left) so in this cell we allot 5 since in this case D > S, we proceed vertically to cell BP.

Step II. In cell BP we allot 2 and since D < we proceed horizontally to cell BQ.

Step III. In cell BQ we can allot only 6 since in this case D > S, proceed vertically to cell CQ.

Step IV. In cell CQ we can allot only 3 and since here D < S, we proceed horizontally to cell CR.

Step V. In cell CR we can allot only 4 since in this cell D < S, proceed vertically to cell DR.

Step VI. In cell DR we can allot 14.

Now, for this solution for transportation cost is

$$Z = \text{Rs. } (2 \times 5 + 3 \times 2 + 6 \times 3 + 3 \times 4 + 7 \times 4 + 2 \times 14)$$
$$= \text{Rs. } (10 + 6 + 18 + 12 + 28 + 28)$$
$$= \text{Rs. } 102.$$

NOTES

It is clear that minimum cost method gives a better solution.

Unbalanced Transportation Problems

Example 6.4. *A departmental store wishes to purchase th following quantity of ladies dresses:*

Dress type	*A*	*B*	*C*	*D*
Quantity	*150*	*100*	*75*	*250*

Tenders are submitted by three different manufacturers who undertake to supply not more than the quantity give below (all types of dresses combined)

Manufacturer	*W*	*X*	*Y*
Total quantity	*350*	*250*	*150*

The store estimates that profit per dress will vary with the manufacturers as shown in the matrix below. How should orders be placed ?

		Dress			
		A	*B*	*C*	*D*
Manufacturer	*W*	*2.75*	*3.50*	*4.25*	*2.25*
	X	*3.00*	*3.25*	*4.50*	*1.75*
	Y	*2.50*	*3.50*	*4.75*	*2.00*

Solution. The problem can be written in the form o the following matrix:

Step I. Matrix

		A	B	C	D	Supply
	W	2.75	3.50	4.25	2.25	300
Manufacturer	X	3.00	3.25	4.50	1.75	250
	Y	2.50	3.50	4.75	2.00	150 (Total 700)
	Demand	150	100	75	250	(Total 575)

Since the supply and demand are not equal, it is not a balanced problem. Here total supply is 700 and total demand is 575, so surplus supplies are 125.

We have to create dummy destination (store). The cost associated with store will be taken zero as the surplus quantity manufactured remains in the factory and is not transported at all, so the new matrix is:

		A	B	C	D	E		Supply
	W	2.75	3.50	4.25	2.25	0		300 (0·25)
Manufacturer	X	3.00	3.25	4.50	1.75	0	125	250 (1.25) ←
	Y	2.50	3.50	4.75	2.00	0		150 (0.50)
	Demand	150	100	75	250	125		
		(0.25)	(0.25)	(0.25)	(0.25)	(0)		

Step II. Using Vogel's Approximation Method (VAM). Let us write the difference between the smallest and second cost in each column and each row and write it below the column or on right side of the rows respectively.

Row with greatest difference is row X as indicated with ¨ an arrow. In this row the least cost cell is XE. In this we can allot 125 since column E is fully satisfied this column is crossed out. ow the shrunken matrix is shown below.

	A	B	C	D	
W	2.75	3.50	4.25	2.25	300 (0.25)
Manufacturer	3.00	3.25	4.50	1.75	
X				(125)	250 (1.25)
Y	2.50	3.50	4.50	2.00	150 (0.50)
Demand	150	100	75	250	
	(0.25)	(0.25)	(0.25)	(0.25)	

Step III. In this matrix maximum difference is in row X and the least cost cell is XD. We can allot 125 units to this cell and since this row is fully satisfied it is crossed out. The new matrix is as follows:

	A	B	C	D	Supply
W	2.75	3.50	4.25	2.25	300 (0.25)
Manufacturer	2.50	3.50	4.75	2.00	150 (0.50)
Y				(125)	
	150	100	75	250	
	(0.25)	(0.25)	(0.25)	(0.25)	

Step IV. In the above matrix maximum difference is in row Y which is shown with an ¨ arrow. In this row least cost cell is YD and so we allot 125 units to this cell sine this satisfies column D so this column is crossed out and the resulting matrix is rewritten as follows:

	A	B	C	
W	2.75 (125)	3.50 (100)	4.25 (75)	300 (0.25)
	150	100	75	
	(0.25)	(0.25)	(0.25)	

Step V. In this least cost cell is WA in which 125 can be allotted. Also in WB we can allot 100 units and in WC 75 can be allotted.

Step VI. The matrix with all allocation is shown below.

	A	B	C	D	E	
W	2.75 (125)	3.50 (100)	4.25 (75)	2.25	0	300
X	3.00	3.25	4.50	1.75 (125)	0 (125)	250
Y	2.50 (25)	3.50	4.75	2.00 (125)	0	150
	150	100	75	250	125	

The cost of this solution is

$$Z = \text{Rs. } (12 \times 2.75 + 100 \times 3.50 + 75 \times 4.25 \times 1.75 + 25 \times 2.50 + 125 \times 2.00)$$

$$= \text{Rs. } (333.75 + 350 + 318.75 + 218.75 + 62.50 + 250)$$

$$= \text{Rs. } 1533.75$$

NOTES

Example 6.5. *The above problem can also be solved with the help of MODI method.*

R_i \ K_j		K_1	K_2	K_3	
	From \ To	A	B	C	Plant capacity
R_1	W	4 (56)	8	8	56
R_2	X	16 (16)	24 (66)	16	82
R_3	Y	8	16 (36)	24 (41)	77
		72	102	41	215

For each square we use the following formula to find its cost:

$$R_i + K_j = C_{ij}$$
$$R_1 + K_1 = 4$$
$$R_2 + K_1 = 16$$
$$R_2 + K_2 = 24$$
$$R_3 + K_2 = 16$$
$$R_3 + K_3 = 24$$
$$R_1 = 0 \text{ then}$$
$$R_1 + K_1 = 4$$
$$0 + K_1 = 4$$
$$K_1 = 4$$

$$R_2 + K_1 = 16 \qquad R_2 + K_2 = 24$$
$$R_2 + 4 = 16 \qquad 12 + K_2 = 24$$
$$R_2 = 12 \qquad K_2 = 12$$

Similarly,

$$R_3 + K_2 = 16 \qquad R_3 + K_3 = 2$$
$$R_3 + 12 = 16 \qquad 4 + K_3 = 24$$
$$R_3 = 4 \qquad K_3 = 20$$

NOTES

R_i \ K_j		$K_1 = 4$	$K_2 = 12$	$K_3 = 20$	
	From \ To	Project A	Project B	Project C	Plant capacity
$R_1 = 0$	W	4 (56)	8	8	56
$R_2 = 12$	X	16 (16)	24 (66)	16	82
$R_3 = 4$	Y	8	16 (36)	24 (41)	77
Requirement		72	102	41	215

Unused squares	$C_{ij} - R_1 - K_1$	Improvement index
$1 \to 2$	$C_{12} - R_1 - K_2$ $8 - 0 - 12$	– 4
$1 \to 3$	$C_{13} - R_1 - K_3$ $8 - 0 - 20$	–12
$2 \to 3$	$C_{23} - R_2 - K_3$ $16 - 12 - 20$	– 16
$3 \to 1$	$C_{31} - R_3 - K_1$ $8 - 4 - 4$	0

∵ The value of water square 23 is most negative, we draw closed loop though this cell and the new table will be

R_i \ K_j		$K_1 = 4$	$K_2 = 12$	$K_3 = 20$	
	From \ To	Project A	Project B	Project C	Plant capacity
$R_1 = 0$	W	4 (56)	8 (– 4)	8 (– 12)	56
$R_2 = 12$	X	16 (16)	24 (66)	16 + (– 16)	82
$R_3 = 4$	Y	8 (0)	16 +(36)	24 (– 41)	77
	Project requirement	72	102	41	215

NOTES

R_i \ K_j		$K_1 = 4$	$K_2 = 12$	$K_3 = 20$	
	From \ To	Project A	Project B	Project C	Plant capacity
$R_1 = 0$	W	4 (56)	8 (– 4)	8 (+ 4)	56
$R_2 = 12$	X	16 (16)	24 (25)	16 (41)	82
$R_3 = 4$	Y	8 (0)	16 (77)	24 (12)	77
	Project requirement	72	102	41	215

Stone square $1 \to 1$

$$R_1 + K_1 = 4$$
$$0 + K_1 = 4$$
$$K_1 = 4$$
$$2 \to 1 \quad R_2 + K_1 = 16$$
$$R_2 + 4 = 16$$
$$R_2 = 12$$
$$2 \to 2 \quad R_2 + K_2 = 24$$
$$12 + K_2 = 24$$
$$K_2 = 12$$
$$2 \to 3 \quad R_2 + K_3 = 16$$
$$12 + K_3 = 16$$
$$K_3 = 4$$
$$3 \to 2 \quad R_3 + K_2 = 16$$
$$R_3 + 12 = 16$$
$$R_3 = 4$$

Calculation of opportunity cost of water squares is as given

Unused squares	$C_{ij} - Ri - K_j$	Improvement index
$1 \to 2$	$C_{12} - R_1 - K_2$ $8 - 0 - 12$	–4
$1 \to 3$	$C_{13} - R_1 - K_3$ $8 - 0 - 4$	+ 4
$3 \to 1$	$C_{31} - R_3 - K_1$ $8 - 4 - 4$	0
$3 \to 3$	$C_{33} - R_3 - K_3$ $24 - 4 - 4$	+ 16

$\therefore$ The opportunity cost of cell $\to 2$ is negative

NOTES

∴ **Third approved solution**

R_i \ K_j		$K_1 = 4$	$K_2 = 8$	$K_3 = 4$	
	From \ To	Project A	Project B	Project C	Plant capacity
$R_1 = 0$	W	4 (31)	8 (25)	8 (+ 4)	56
$R_2 = 12$	X	16 (41)	24 (4)	16 (41)	82
$R_3 = 8$	Y	8 (– 4)	16 (77)	24 (12)	77
		72	102	41	215

∵ The value of cell → 1 is negative in the improved solution.

R_i \ K_j		$K_1 = 0$	$K_2 = 8$	$K_3 = 0$	
	From \ To	A	B	C	Plant capacity
$R_1 = 0$	W	4 (+ 4)	8 (56)	8 (8)	56
$R_2 = 16$	X	16 (41)	24 (0)	16 (41)	82
$R_3 = 8$	Y	8 (31)	16 (46)	24 (16)	77
	Project requirement	72	102	41	215

Total cost of optimal solution

Shipping assignment	**Quantities shipped**	**Limit cost**	**Total cost**
WB	56	8	448
XA	41	16	656
XC	41	16	656
YA	31	8	248
YB	46	16	736
			2744

Example 6.6. *ABC tool company has a sales force of 25 men who work out from three regional centres. The company produces four basic product lines of hand tools. Mr. Jain, Sales Manager feels that 6 salesmen are needed to distribute product line 1, 10 salesmen to distribute product line 2, 4 salemen to product line 3 and 5 salesmen to product line 4. The cost (in Rs.) per day of assigning salesmen for each of the offices for sellingeach of the product lines are as follows :*

NOTES

Regional Office	Product Lines 1	2	3	4
A	*20*	*21*	*16*	*18*
B	*17*	*28*	*14*	*16*
C	*29*	*23*	*19*	*20*

At the present time, 10 salesmen are allowed to office A, 9 salesmen to office B and 7 salesmen to office C. How many salesmen should be assigned from each office to each product line in order to minimize costs ?

Solution Initial Feasible Solution is as follows:

From \ To	1	2	3	4	5 Dummy	Availability
A	20	21 (4)	16 (1)	18 (5)	0	10
B	17 (6)	28	14 (3)	16	0	9
C	29	23 (6)	19	20	0 (1)	7
Requirement	6	10	4	5	1	26

Let us now apply MODI method to est the optimality of the above solution :

	R_i \ K_j		K_1 19	K_2 21	K_3 16	K_5 18	K_6 – 2	
		From \ To	1	2	3	4	5 Dummy	Availability
R_1	0	A	20 / +1	21 / 4	16 / 1	18 / 5	0 / +2	10
R_2	– 2	B	17 / 6	28 / +9	14 / 3	16 / 0	0 / +4	9
R_3	2	C	29 / +8	23 / +6	19 / +1	20 / 0	0 / 1	7
	Requirement		6	10	4	5	1	26

The above is the optimum solution, since all optimality costs are positive.

Total cost = 21 × 4 + 16 × 1 + 18 × 5 + 17 × 6 + 14 × 3 + 23 × 6 + 0 × 1 = Rs. 472

Degeneracy in the Transportation Problem

We have seen that an initial feasible solution to an *m* resources/origins and *n* destination problem consists of $(m + n – 1)$ basic variables which is the same as the number of occupied cells. However, if the number of occupied cells is less than $(m + n – 1)$ at any stage of the solution, then the transportation problem is said to have a degenerate solution. Degeneracy as it is called can occur at two stages, *i.e.*, at the initial solution or during the testing of the optimal solution. Let us discuss both the cases.

Degeneracy at the Initial Solution Stage

If degeneracy occurs at the initial solution stage, we introduce a very small quantity ε (Greek letter pronounced as epsilon) in one or more of the unoccupied cells to make the number of occupied cells equal to $(m + n - 1)$. ε is so small a quantity that its introduction does not change the supply (sources) and demand (destinations) constraints or the rim conditions. ε is placed in the unoccupied cell which has the least transportation cost and once ε is allotted to it, it is supposed to have been occupied. ε stays in the solution till degeneracy is removed or the final solution is achieved. The value of ε is zero when used in the problems with movement of goods from one cell to another.

The use of ε and degeneracy can be explained with the help of examples. Here degeneracy occurs at the initial solution.

6.11 SUMMARY

- The transportation model seeks the determination of a transportation plan of a single commodity from a number of sources to a number of destinations. The model must have the following information:
  (a) Amount of demand at each destination
  (b) Availability at each source
  (c) The unit transportation cost of commodity from each source to each destination.

  Since we are concerned with only one commodity, the destination can get the commodity from any of the sources. The objective of the problems is to find out the amount (quantity) to be transported from each of the sources to each destination so that the total transportation cost is minimum.
- *Feasible solution*: Non-negative values of x_{ij} where $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$ which satisfy the constraints of availability (supply) and requirement (demand) is called the feasible solution to the transportation problem
- *Basic feasible solution:* It is the feasible solution that contains only $m + n - 1$ non-negative allocation.
- *Optimal solution:* A feasible solution is said to be optimal solution when the transportation cost is minimum.
- *Balanced transportation problem:* A transportation problem in which the total supply from all the sources equals the total demand in all the destinations.
- *Unbalanced transportation problem:* Such problems which are not balanced are called *unbalanced.*
- *Matrix terminology:* In the matrix used in transportation problem, the squares are called *cells*. These cells form ‘columns’ vertically and ‘rows’ horizontally. Unit costs are written i the cells.
- *Availability of commodity:* The supply available at different sources is equal to or more than the total demand of different destinations when it is equal it is called a *balanced problem*
- *Transportation of commodity/items:* The model assumes that all items can be conveniently transported from sources to destinations.
- *Certainty of per unit transportation cost:* There is a definite cost of transportation of items from sources to destinations.

NOTES

- *Independent cost per unit:* The per unit cost is independent of the quantity transported from sources to destinations.
- *Transportation cost:* Transportation cost on any given route is proportional to the number of units transported.
- *Objective function:* The objective is to minimize the total transportation cost for the entire organization.

6.12 REVIEW AND DISCUSSION QUESTIONS

1. What is a transportation problem? How is it useful in business and industry?

2. Explain the use of transportation problem in business and industry giving suitable examples.

3. What do you understand by

(*a*) Feasible solution;

(*b*) North-West solution;

(*c*) Vogel's Approximation Method (VAM)?

4. Discuss various steps involved in finding initial feasible solution of a transportation problem.

5. Discuss any two methods of solving a transportation problem. State the advantages and disadvantages of these methods.

6. How can an unbalanced transportation problem be balanced ? How do you interpret the optimal solution of an unbalanced transportation problem ?

7. Explain the differences and similarities between the MODI method and stepping stone method used for solving transportation problems.

8. What is a transportation method? Explain its objectives. How can we used this model for solving a multiple-site facility beaten problem ?

9. Which method of solving transportation problems gives a more optimal solution ? How will you know when you have achieve the least cost allocation of products between origins and destinations? Explain with examples.

10. Formulate a cost minimization model for the allocation of facilities to locations in the following problem:

ABC Ltd. is considering the layout of one of its plants divided into three different working areas. There are three different production facilities and each one has one of them. Assume the data not available.

11. Plant location of a firm manufacturing a single product has three plants located at A, B and C. Their production during week has been 60, 40 and 50 units respectively. The company has firm commitment orders for 25, 40, 20, 20 and 30 units of the product to customers C–1, C–2, C–3, C–4 and C–5 respectively. Unit cost of transporting from the three plants to the five customers is given in the table below.

		C–1	C–2	C–3	C–4	C–5
	A	6	1	3	4	6
Plant location	B	4	4	3	3	2
	C	2	6	4	4	6

Use VAM to determine the cost of shipping the product from plant locations to the customers.

12. Solve the following transportation problem. Availability at each plant, requirements at each warehouse and the cost matrix is as shown below.

		Warehouse				
		W_1	W_2	W_3	W_4	Availability
	P_1	200	400	600	200	80
Plant	P_2	800	400	400	500	100
	P_3	400	200	500	400	190
	Requirement	60	80	80	120	

13. There are four supply points P–1, P–2, P–3 and P–4 5 / destination A, B, C, D and E. The following table gives in cost of transportation of materials from supply points to demand statements in rupees.

		To :				
		A	B	C	D	E
	P–1	10	12	15	16	18
	P–2	12	10	12	10	10
From :	P–3	15	20	6	12	16
	P–4	12	18	10	12	12

The present allocation is as follows:

P–1 to A 100, P–1 to B–20, P–2 to B–160, P–3 to B–10, P–3 to C–60, P–3 to E–120, P–4 to D–200, P–4 to E–100. Find an optimal solution for allocations. If we reduce the cost from any supply point to any destination, what do you think will be the impact. Select any case and discuss the outcome.

14. A steel company has three furnaces and five rolling mills. Transportation cost (rupees per quintal) for sending steel from furnaces to rolling mills are given in the following table :

Furnaces	M_1	M_2	M_3	M_4	M_5	**Availability (Q)**
A	4	2	3	2	6	8
B	5	4	5	2	1	12
C	6	5	4	7	3	14
Requirement (Quintal)	4	4	10	8	8	

How should they meet the requirement ? Use VAM.

15. A cement factory manager is considering the best way to transport cement from his three and manufacturing centres P, Q and R to depot A, B, C, D and E. The weekly production and demand along with transportation costs per ton are given below.

	A	B	C	D	E	**Tons**
P	4	1	3	4	4	60
Q	2	3	2	4	3	35
R	3	5	2	2	4	40
Requirement (Quintal)	22	45	20	18	30	135

What should be the distribution programme ?

NOTES

16. The cost conscious company requires for the next month 300, 260 and 180 tones of stone chips for its three constructions, C_1, C_2 and C3 respectively. Stone chips are produced by the company at three mineral fields taken on short lease. All the available boulders must be curshed into chips. Any excess chips over the demands at sites C_1, C_2 and C_3 will be sold ex-fields. The fields M_1, M_2 and M_3 will yield 250, 320 and 280 tones chips respectively. Transportation costs from mineral fields to construction sites vary according to distances, which are given below in monetary unit (MU).

From \ To	C_1	C_2	C_3
M_1	8	7	6
M_2	5	4	9
M_3	7	5	5

(*i*) Determine the optimal economic transportation plan for the company and the overall transportation cost in MU.

(*ii*) What are the quantities to be sold from M_1, M_2 and M_3 respectively ?

17. A company has 4 different factories in 4 different locations in the country and four sales agencies in four other locations in the country. The cost of productions, the sale price, shipping cost in the cell of matrix, monthly capacities and monthly requirements are given below.

Sale Agency

Factory	1	2	3	4	Capacity	Cost of production
A	7	5	6	4	10	10
B	3	5	4	2	15	15
C	4	6	4	5	20	16
D	8	7	6	5	15	15
Monthly requirements	8	12	18	22		
Selling price	20	22	25	18		

Find the monthly production and distribution schedule, which will maximize profits.

18. A leading firm has three auditors. Each auditor can work upto 160 hours during the next month, during which time three projects must be completed. Project 1 will take 130 hours.

19. Determine an initial basic feasible solution to the following transportation problem using North-West Corner Rule:

	To :					Available
From :	3	4	6	8	9	20
	2	10	1	5	8	30
	7	11	20	40	3	15
	2	1	9	14	16	13
Demand	40	6	8	18	6	

UNIT 7: ASSIGNMENT MODEL

NOTES

Structure

7.1 INTRODUCTION

In real life situations, problems arise where a number of resources have to be allotted to a number of activities. In a sense, a special case of the transportation model is the Assignment Model. This model is used when the resources, have to be assigned to the tasks, i.e., assign *n* persons to *n* different type of jobs. Since different types of resources whether human, i.e., men or material, machines, etc., have different efficiency of performing different types of jobs and it involves different costs, the problem is how to assign such resources to jobs so that total cost is minimized or given objective is optimized. A plant may have 10 persons and 10 different types of job, the plant manager would like to know which person should be allotted which job so that all the jobs can be completed in least time (and hence least cost). Similarly, if a transporter has six trucks available for loading in each of the cities A, B, C, D, E and F and it actually needs these trucks in six locations 1, 2, 3, 4, 5 and 6, obviously the trucker would like to know which truck should be assigned to which location so that the transportation costs are minimized. In the same manner if a sales agency has say four sales man available (with different abilities and perhaps different capacities) and there are four territories where the agency wants to assign these sales man, the problem is which salesman should be allotted to which territory so as overall sales can be maximized.

An assignment problem is, in fact, a completely degenerate form of a transportation problem. In this case, the units (resources) available at each origin and units demanded at each destination are all equal to one, *i.e.*, exactly one occupied cell in each row and each column of the transportation table.

7.2 DEFINITION OF ASSIGNMENT MODEL

Let us consider an assignment problem involving *n* resources (origins) to *n* destinations. The objective in making the assignment can be one of minimization or maximization (*i.e.*, minimization of total time required to complete *n* tasks or maximization of total profit from assigning salespersons to sales territories). The following assumptions have to be made while formulating assignment models:

Assumptions

NOTES

1. Each resource is assigned exclusively to one task.
2. Each task is assigned exactly to one resource.
3. For purposes of solution, the number of resources available for assignment must equal the number of tasks to be performed. Let x_{ij} be the variable in such a way that if

$$x_{ij} = \begin{bmatrix} 1 \text{ if resource } i \text{ is assigned to task } j \\ 0 \text{ if resource } i \text{ is not assigned to task } j \end{bmatrix}$$

C_{ij} = objective function contribution if resource i is assigned to task j.

n = number of resources and number of tasks.

Clearly, since only one job is to be assigned to each resource.

$$\sum_{i=1}^{n} x_{ij} = 1 \quad \text{and} \quad \sum_{j=1}^{n} x_{ij} = 1$$

And the total assignment cost will be given by

$$Z = \sum_{j=1}^{n} \sum_{i=1}^{n} x_{ij} C_{ij}$$

Hence, the mathematical formulation assumes the following form:

Determine $x_{ij} \geq 0$ $(i, j = 1, 2, \ldots\ldots\ldots n)$ so as to minimize $Z = \sum_{j=1}^{n} \sum_{i=1}^{n} x_{ij} C_{ij}$

Subject to the following constraints :

$$\sum_{i=1}^{n} x_{ij} = 1 \qquad j = 1, 2, 3, \ldots\ldots\ldots, n$$

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad i = 1, 2, 3, \ldots\ldots, n$$

and $\quad x_{ij} = 0$ or 1

The general assignment model can be written as

Maximize (or minimize) $Z = C_{11}\, x_{11} + C_{12}\, x_{12} + \ldots\ldots\ldots\ldots + C_{1n}\, x_{1n} + C_{21}\, x_{21} + \ldots\ldots\ldots + C_{nn}\, x_{nn}$ subject to

$$\begin{aligned}
x_{11} + x_{12} + \ldots\ldots\ldots + x_{1n} &= 1 \\
x_{21} + x_{22} + \ldots\ldots\ldots + x_{2n} &= 1 \\
&\vdots \\
&\vdots \\
x_{n1} + x_{12} + \ldots\ldots\ldots\ldots + x_{nn} &= 1 \\
x_{11} + x_{21} + \ldots\ldots + x_{n1} &= 1 \\
x_{12} + x_{22} + \ldots\ldots + x_{n2} &= 1 \\
\vdots \quad \vdots \quad \vdots \\
\vdots \quad \vdots \quad \vdots \\
x_{1n} + x_{2n} + \ldots\ldots + x_{nn} &= 1
\end{aligned}$$

$x_{ij} = 0$ or 1 for all values of i and j.

The student should notice that for this model the variables are restricted to the two values i.e., 0 for non –assignment of the resources or 1 for assignment of the resource. This restriction is quite different from the other Linear Programming models we have seen so far.

Theorem 1. The optimum assignment schedule remains unaltered if we add or subtract a constant to/from all elements of the row or column of the assignment cost matrix.

Theorem 2. If for an assignment problem all $C_{ij} \geq 0$ then an assignment schedule (x_{ij}) which satisfies $\Sigma\,\Sigma\, x_{ij}\, C_{ij} = 0$ must be optimal. These two theorems are the basis of the assignment algorithm. We add or subtract suitable constant to/from the elements of cost matrix in such a way that new $C_{ij} \geq 0$ and can produce at least one new $C_{ij} = 0$ in each row and each column and try and make assignments from among these 0 positions. The assignment schedule will be optimal if there is exactly one assignment in each row and each column (*i.e.*, exactly one assigned 0).

Solution of Assignment Problems

1. Complete Enumeration Method

In this method costs for all possible assignments are worked out and the one having the minimum cost is termed as the optimal solution. This method, for obvious reasons, can only be used for small problems. As the problems become complex, it is impractical to workout a very large number of alternatives and then pick up the optimal solution.

2. Simplex Method

In this method the simplex algorithm is used and we

Minimize or Maximize $Z = \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij}\; x_{ij}$

Subject to constraints

(*i*) $x_{i1} + x_{i2} + \ldots\ldots\ldots + x_{in} = 1$ $\qquad i = 1, 2, \ldots\ldots, n$

(*ii*) $x_{1j} + x_{2j} + \ldots\ldots\ldots + x_{nj} = 1$ $\qquad j = 1, 2, \ldots\ldots, n$

(*iii*) $x_{ij} = 0$ or 1 for all values of i and j.

It can be seen there are $n \times n$ decision variables and $n + n = 2n$ equalities. It means that for a problem involving 8 workers/jobs there will be 64 decision variables and 16 equalities to be solved. It is an extremely cumbersome method.

3. *Transportation Method*

We have earlier mentioned that assignment model is a special case of transportation model, so it should be possible to solve it by transportation method. However, we know that optimality test in the transportation method requires that there should be $n + n - 1 = 2n - 1$ basic variables, the solution obtained by this method would be severally degenerate. For an assignment made there would be only n basic variables in the solution, hence to proceed for solving an assignment model by using transportation model, a very large number of dummy allocations will have to be made, which will make this method very inefficient to compute.

4. *Hungarian Assignment Method or HAM (Minimization case)*

This method was developed by Hungarian mathematician D Koning and is also known as the *Flood's Technique* or the *reduced matrix method*. It is a simpler and more efficient method of solving the assignment problems. The following steps are involved while using the Hungarian method:

NOTES

Step I. Formulate the opportunity cost table by the following method :

(*a*) Subtract the smallest number in each row of the original cost matrix from every number in that row.

(*b*) Subtract the smallest number in each column of the table obtained at (a) above from every number in that column.

Step II. Make assignments in the following manner :

(*a*) Examine all the rows looking for a row with exactly one unmarked zero in a as square (□) as assignment has to made there. Cross (×) all other zeros in the column as these will not get an assignment in future. Proceed in this manner for all the rows.

(*b*) Now examine all the columns one by one until we find a column with exactly one marked zero is located. Make an assignment to this single zero and pt a square (□) around it. Also cross out (×) all other zeroes appearing in the corresponding row as no assignment will be made in that row. Proceed in this manner for all the columns.

(c) Operations (a) and (b) are repeated till.

(*i*) All the zeros in rows/columns are either put in the square (□) or are crossed out (×) and exactly one assignment is in each ro each column. This is the optimal solution.

(*ii*) Some row or column may be left without assignment, if so proceed to step III.

Step III. Revise the opportunity cost matrix by

(*i*) Marking (√) all rows that have no assignment.

(*ii*) Marking (√) all columns which have zeros but have not been marked earlier.

(*iii*) Marking (√) all rows that have assignments and have not been marked earlier.

(*iv*) Repeat step III (*i*) and (*ii*) until no more rows and columns can be marked.

(*v*) Draw straight lines through each unmarked row and each marked column.

Now check the total assignments indicated by number of lines drawn is equal to the number of rows or columns, the optimal solution has been reached. Otherwise proceed to step IV.

Step IV. Write the new revised opportunity cost matrix.

Initial opportunity cost matrix may never give the optimal solution, we are normally required to revise this table in order to move one or more zero costs from present location to new uncovered locations. This is done by subtracting the smallest number not covered by a line from all numbers not covered by a straight line. This number (smallest) is added to every number, including zeros available at the intersection of any two lines.

Step V. Repeat steps II to IV until an optimal solution is achieved.

The above steps are shown in the form of a flow chart in the following figure:

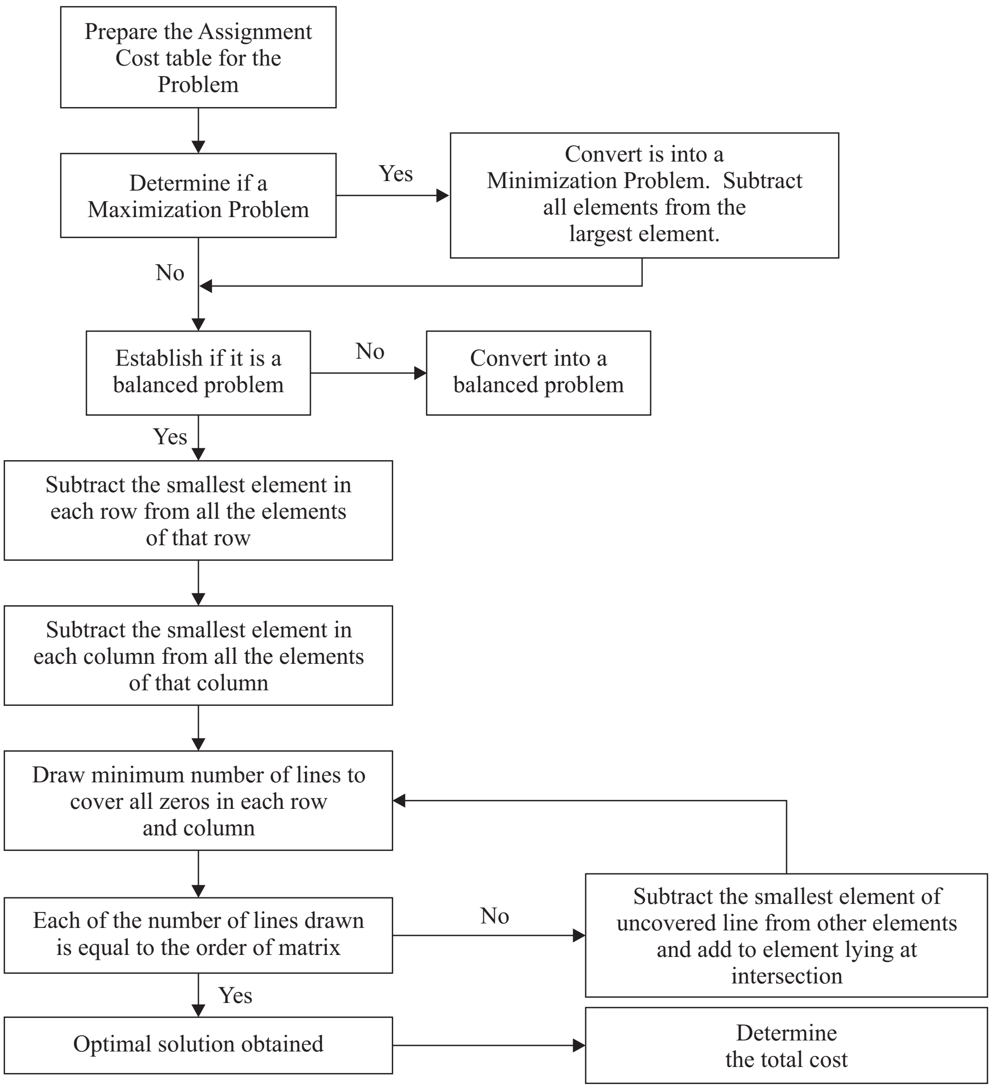


Fig. 6.1

7.3 PRACTICAL STEPS INVOLVED IN SOLVING MINIMIZATION / MAXIMIZATION PROBLEMS

Minimization Problems

Step I. Check if the number of rows is equal to number of columns. If it is equal, the problem is a balanced one. If not, add a dummy column or row to make it a balanced one, by allotting zero values to each element (cell) of dummy row or column as the case is.

Step II. Row subtraction. Subtract the minimum or least value element of each row from all elements of that row.

Step III. Column subtraction. Subtract the minimum or least value element of each column from all elements of that column.

NOTES

Step IV. Draw minimum number of horizontal and /or vertical lines in such a manner that all zeros are covered. For doing this, follow the procedure as follows :

(a) Select a row containing exactly one uncovered zero and draw a vertical line through the column containing this zero and repeat the process till no such row is left.

(b) Select a column containing exactly one uncovered zero and draw a horizontal line through the row containing the zero and repeat the process till no such column is left.

Step V. If the total number of minimum lines covering all zeros is equal to the order of the matrix, then we have got the optimal solution and there is no need to proceed further.

Step VI. If not, subtract the minimum uncovered element from all the uncovered elements and add this element to all the elements at the intersection points of the lines covering zeros.

Step VII. Repeat steps IV, V and VI till minimum number of lines covering all zeros is equal to the order of the matrix.

Step VIII. Now make assignments by selecting a row containing exactly one unmarked zero and put a square □ around it. Also, draw a vertical line through the column containing this zero. Repeat this rocess till no such row is left. Move to the column containing exactly one unmarked zero and put a square □ around it, also, draw a horizontal line through the row containing this zero and repeat this process till no such column is left. If there are more than one zeros in any one row or column, select any one arbitrarily and pass two lines horizontally and vertically.

Step IX. Find the optimum value by adding up the values of the final assignments.

Maximization Problems

Step I. Rewrite the problem as a minimization problem by subtracting all elements from the largest element.

Step II. Follow the same steps as above from I to IX.

Unbalanced Problems

Establish whether the problem is a balanced one, i.e., the number of rows = number of columns. If yes proceed as discussed earlier. If not, then add a dummy row or column to make the problem a balanced one by allotting zero values to each cell of the dummy row or column, as the case may be.

Example 7.1. *A department head has four subordinates and four tasks to be performed. Subordinates differ in efficiency and the tasks differ in their intrinsic difficulty. His estimate of time each man would take to perform each task is given in the matrix below.*

Tasks	*Men*			
	E	*F*	*G*	*H*
A	*18*	*26*	*17*	*11*
B	*13*	*28*	*14*	*26*
C	*38*	*19*	*18*	*15*
D	*19*	*26*	*24*	*10*

How should the tasks be allotted, one to a man, so as to minimize the total man hours ?

Solution.

Step I. Subtract the smallest element of each row from every element of the corresponding row, the reduced matrix is as follows (subtract 11 from row one, 13 from row 2, 15 from row 3 and 0 from row 4)

NOTES

$$\begin{bmatrix} 7 & 15 & 6 & 0 \\ 0 & 15 & 1 & 3 \\ 23 & 4 & 3 & 0 \\ 9 & 16 & 14 & 0 \end{bmatrix}$$

Step II. Subtract the smallest element of each column of the above-reduced matrix from every element of the corresponding column, we get the following reduced matrix (subtract 0 from column 1,4 from column 2,1 from column 3 and 0 fom column 4),

$$\begin{bmatrix} 7 & 11 & 5 & 0 \\ 0 & 11 & 0 & 13 \\ 23 & 0 & 2 & 0 \\ 9 & 12 & 13 & 0 \end{bmatrix}$$

Step III. Starting with row 1, we make assignment to a single zero an put a square $(\square)$ around it and cross out all other zeros in the column so marked. By doing so, we get

$$\begin{bmatrix} 7 & 11 & 5 & \boxed{0} \\ \boxed{0} & 11 & \not{0} & 13 \\ 23 & \boxed{0} & 2 & \not{0} \\ 9 & 12 & 13 & \not{0} \end{bmatrix}$$

Note that row 2 had two zeros, we have arbitrarily made assignment to zero in column 1 and put a square around it. Also, note that column 3 and row 4 do not have any assignment.

Step IV. (*i*) Tick row 4 as it does not have any assignment. In fourth column of the ticked row (row 4), there is a zero, so we tick fourth column.

(*ii*) In the first row of ticked column (column 4) there is an assignment made, so first row is ticked.

(*iii*) Draw straight lines through all unmarked rows and marked columns.

The following matrix shows the above operations:

$$\begin{bmatrix} 7 & 11 & 5 & \boxed{0} \\ \boxed{0} & 11 & \not{0} & 13 \\ 23 & \boxed{0} & 2 & \not{0} \\ 9 & 12 & 13 & \not{0} \end{bmatrix} \begin{matrix} \checkmark \\ \\ \\ \checkmark \end{matrix}$$

Step V. Is the present solution an optimal solution ? To find that we know, there are 3 lines drawn which is less than the order of the cost matrix (4). Hence it is not an optimal solution we have to generate new zeros in the matrix to increase the minimum number of lines.

NOTES

Step VI. Let us find out the smallest element not covered by the lines. It is 5. Subtract 5 from all the uncovered elements and add 5 to all the elements lying at the intersection of lines, we get the following reduced matrix.

$$\begin{bmatrix} 2 & 6 & 0 & 0 \\ 0 & 11 & 0 & 18 \\ 23 & 0 & 2 & 5 \\ 4 & 7 & 8 & 0 \end{bmatrix}$$

Step VII. Repeat step III on the reduced matrix, i.e., scrutinise all the elements row wise, say with row one, make assignment to a single zero and crossout all other zeros in the column so marked, we get

	E	F	G	H
A	2	6	[0]	~~0~~
B	[0]	11	~~0~~	18
C	23	[0]	2	5
D	4	7	8	[0]

It can be seen now each row and each column has only one assignment, an optimal solution has been obtained. The optimum assignment is A → G, B → E, C → F, D → H.

The minimum total time for this assignment would be obtained by adding the relevant times, i.e., 17 + 13 + 19 + 10 = 59 man-hours.

The Travelling Salesman Problem

One of the major applications of the assignment models is in the travelling salesman problem.

Let us say that a salesman has to visit n destinations. He starts from a particular city, visits each destination once and then comes back to the city from where he started. Here the objective would be to minimise the time this salesman takes to visit all the destinations. The problem is to select that sequence in which all the destinations are visited in such a manner that the time taken is minimised. This problem is similar to the assignment problems already seen except that there is an additional restriction that the salesman starting from a particular city, visits each destinations only once and returns to his city from where he started.

Formulation of Travelling Salesman Problem

Let us define the variables x_{ijk} as (notice constraint k has been added in addition to i and j already here)

$$x_{ijk} = \begin{bmatrix} 1 & \text{if } k\text{th destination from } i \text{ to } j \\ 0 & \text{if } k\text{th destination is not from city } i \text{ to city } j \end{bmatrix}$$

where i, j and k are integers which vary between 1 and n. The following constraints can be put in the mathematical for as shown.

$$(a) \quad \sum_{j}^{n} \sum_{k}^{n} x_{ijk} = 1, \quad k = 1, 2, 3, \ldots\ldots n \text{ and } i \neq j$$

NOTES

(*b*) As only one city can be reached from a specific city, say i,

$$\sum_{j}^{n} \sum_{k}^{n} x_{ijk} = 1 \quad \text{where} \quad i = 1, 2, 3, \ldots\ldots n$$

(*c*) And only from one city the salesman can go to a specific city say j

$$\sum_{i}^{n} \sum_{k}^{n} x_{ijk} = 1 \quad \text{where} \quad j = 1, 2, 3, \ldots\ldots n$$

(*d*) Given that kth visit ends at some specific city j, $(k + 1)$ th visit must start at the same city j thus we have

$$\sum_{i \neq j}^{n} x_{ijk} = \sum_{i \neq j}^{n} x_{ij(k+1)} \text{ for all values of } j \text{ and } k.$$

Now the objective function

$$\text{Minimise } Z = \sum_{i}^{n} \sum_{j}^{n} \sum_{k}^{n} x_{ijk}\, d_{ij} \qquad i \pi j$$

where d_{ij} is the distance from city i to city j.

Example 7.2. *Solve the following travelling salesman problem so as to minimize the cost per cycle:*

From	A	B	C	D	E
A	–	3	6	2	3
B	3	–	5	2	3
C	6	5	–	6	4
D	2	2	6	–	6
E	3	3	4	6	–

Solution. Step I. Row Reduction : let us subtract the smallest element of each row from other elements of the same row. In prohibited cell ∞ will be assigned is shown below :

To

From	A	B	C	D	E
A	∞	1	4	0	1
B	1	∞	3	0	1
C	2	1	∞	2	0
D	0	0	4	∞	4
E	0	0	1	3	∞

Step II. Column Reduction : Subtract the smallest element of each column from other elements of the same column. Assignments will be made in rows and columns having single zero.

To

From	A	B	C	D	E
A	∞	1	3	0	1
B	1	∞	2	0	1

C	2	1	∞	2	0
D	0	0	3	∞	4
E	0	0	0	3	∞

NOTES

Step III. Draw the minimum number of lines to cover all the zeros after marking rows and columns and after drawing lines through unmarked rows and marked columns. This is shown below.

To

From	A	B	C	D	E
A	∞	1	3	[0]	1
B	1	∞	2	~~0~~	1
C	2	1	∞	2	[0]
D	[0]	~~0~~	3	∞	4
E	~~0~~	~~0~~	[0]	3	∞

Step IV. The above table can be modified by subtracting the lowest element, *i.e.*, 1, from all the elements not yet covered by the lines and adding the same in the intersection of the two lines. Assignments can now be made. This gives the table as follows.

To

From	A	B	C	D	E
A	∞	[0]	2	~~0~~	~~0~~
B	~~0~~	∞	1	[0]	~~0~~
C	2	1	∞	3	[0]
D	[0]	~~0~~	3	∞	4
E	~~0~~	~~0~~	[0]	4	∞

Step V. The above table gives an infeasible solution as the optimum assignment is A → B, B → D, D → A and the salesman does not visit C and E and returns home to A.

Let us try and find the next best solution in which C and E cities are also visited by the salesman.

Step VI. Let us make assignment at (2, 3) instead of 0 zero at (2, 4), which is the next higher value, 1 in the matrix. Assign 1

The table i as shown below.

To

From	A	B	C	D	E
A	∞	~~0~~	2	[0]	0
B	~~0~~	∞	[1]	~~0~~	0
C	2	~~1~~	∞	3	[0]
D	~~0~~	[0]	3	∞	4
E	[0]	~~0~~	~~0~~	4	∞

It may be seen optimum assignment is

A → D → B → C → E → A

Minimum cost per cycle = 2 + 2 + 5 + 4 + 3 = 16.

NOTES

7.4 UNBALANCED PROBLEMS

Example 7.3. *A transport corporation has three vehicles in three cities. Each of vehicles can be assigned to any of the four other cities. The distance differs from one city to the other as under.*

	1	*2*	*3*	*4*
A	*33*	*40*	*43*	*32*
B	*42*	*30*	*31*	*24*
C	*40*	*31*	*37*	*31*

You are required :

(a) To assign a vehicle to a city in such a way that the total distance travelled is minimized.

(b) Formulate a mathematical model of the problem.

Solution. (*a*) Step I. Introduce a dummy vehicle in city D as the problem is not balanced and take 0 distances as follows:

	1	2	3	4
A	33	40	43	32
B	42	30	31	24
C	40	31	37	31
D	0	0	0	0

Step II. Subtract the minimum element of each row from all the elements of that row. There is no need to subtract 0 from each column. The matrix is

1	8	11	0
18	6	7	0
9	0	6	0
0	0	0	0

Step III. Draw minimum number of lines to cover all zeros. It can be seen that number of lines = 3, order of matrix = 4. Hence this is not an optimal solution and we have to take steps to increase number of zeros.

1	8	11	0
18	6	7	0
9	0	6	0
0	0	0	0

Step IV. Subtract the minimum uncovered element (1) from all the uncovered elements and adding it to the elements at the intersection point. Draw minimum number of lines to cover all the zeros, we get

~~0~~	~~7~~	~~10~~	~~0~~
17	5	6	0

9	0	6	1
0	0	0	1

NOTES

Now, number of lines = 4, and is equal to the order of the matrix hence, this matrix will provide an optimal solution.

Step V. Assignments can be made as shown below.

[0]	7	10	0
17	5	6	[0]
9	[0]	6	1
0	0	[0]	1

Step VI. Minimum distance can be worked out as follows :

City	Vehicle	Distance
A	1	33
B	4	24
C	2	31
D	3	0
		Total = 88

(*b*) Formulation of LPP.

33 x_{11}	40 x_{12}	43 x_{13}	32 x_{14}
42 x_{21}	30 x_{22}	31 x_{23}	24 x_{24}
40 x_{31}	31 x_{32}	37 x_{33}	31 x_{34}
0 x_{41}	0 x_{42}	0 x_{43}	0 x_{44}

Minimize $Z = 33\,x_{11} + 40\,x_{12} + 43\,x_{13} + 32\,x_{14} + 42\,x_{21} + 30\,x_{22} + 31\,x_{23} + 24\,x_{24} + 40\,x_{31} + 31\,x_{32} + 37\,x_{33} + 31\,x_{34} + 0\,x_{41} + 0\,x_{42} + 0\,x_{43} + 0\,x_{44}$

Subject to $x_{11} + x_{12} + x_{13} + x_{14} = 1$

$x_{21} + x_{22} + x_{23} + x_{24} = 1$

$x_{31} + x_{32} + x_{33} + x_{34} = 1$

$x_{41} + x_{42} + x_{43} + x_{44} = 1$

$x_{ij} \geq 0.$

Example 7.4. *A Company is faced with assigning 5 jobs to 5 operators. Each job must be performed only by one operator. The cost of processing of each job by each operator is given below in Rs.*

Operators

	P	Q	R	S	T
A	7	5	9	8	11
B	9	12	6	11	10

NOTES

	Jobs					
	C	8	5	4	6	8
	D	7	3	6	8	5
	E	5	6	7	5	11

Determine the assignment of jobs to the operators so that it will result in minimum cost.

Solution. Step I. Select the minimum element in each row and subtract this element from every other element in that row.

	P	Q	R	S	T
A	2	0	4	3	6
B	3	6	0	5	4
C	4	1	0	2	4
D	4	0	3	5	2
E	0	1	2	0	6

Step II. Now select the minimum element in each column and subtract this element from every element of that column, we get the following matrix :

	P	Q	R	S	T
A	2	0	4	3	6
B	3	6	0	5	2
C	4	1	0	2	2
D	4	0	3	5	0
E	0	1	2	0	4

Step III. In row A, there is a single zero so assignment is made in cell AQ. In row B, there is a single zero, so assignment is made in cell BR. While making assignment in row AQ, the other zero appearing in the column Q, *i.e.*, in element DQ is crossed out. Similarly, when assignment is made in row B in cell BR, the other zero in column R, *i.e.*, in cell R is crossed out.

	P	Q	R	S	T
A	2	[0]	4	3	6
B	3	6	[0]	5	2
C	4	1	~~0~~	2	2
D	4	~~0~~	3	5	[0]
E	[0]	1	2	0	4

While inspecting rows, row D has a single zero, assignment can be made in cell DT. No assignment can be made in row E, since it has two zeros.

Now inspect columns, column P has single zero, so assignment can be made in cell EP and other zero in row E, *i.e.*, in cell ES can be crossed out.

Since it is possible only to make 4 assignments against 5, so optimum solution is not reached. Number of lines drawn is equal to the number of assignments made.

NOTES

Step IV. Examine the elements not covered by the lines and select the smallest element, in this case 2 and subtract this from all the uncovered elements and add this to the elements lying at the intersection of the two lines. This gives us the following matrix :

	P	Q	R	S	T
A	0	[0]	4	1	4
B	1	6	[0]	3	2
C	2	1	~~0~~	[0]	2
D	2	~~0~~	3	3	[0]
E	[0]	3	4	0	6

Step V. Make assignment in row C where there is a single zero. Hence assignment in cell CS is made.

Step VI. Optimum solution

$A \rightarrow Q = 5$

$B \rightarrow R = 6$

$C \rightarrow S = 6$

$D \rightarrow T = 5$

$E \rightarrow P = 5$

Minimum Total Cost = Rs 27.

Example 7.5. *Five lathes are to be allotted to five operators, the following table gives weekly output figures:*

		L_1	L_2	L_3	L_4	L_5
Operator	P	20	22	27	32	36
	Q	19	23	29	34	40
	R	23	28	35	39	34
	S	21	24	31	37	42
	T	24	28	31	36	41

Profit per piece is 25%. Find the maximum profit per week.

Solution. As the given problem is a maximization problem, we convert it into an opportunity loss matrix by subtracting all the elements of the given table from the highest element of table, *i.e.,* 42. Opportunity loss matrx is as follows:

NOTES

		L_1	L_2	L_3	L_4	L_5
Operator	P	22	20	15	10	6
	Q	23	19	13	8	2
	R	19	14	7	3	8
	S	21	18	1	5	0
	T	18	14	11	6	1

Row Reduction

		L_1	L_2	L_3	L_4	L_5
Operator	P	16	14	9	4	0
	Q	21	17	11	6	0
	R	16	11	4	0	5
	S	21	18	11	5	0
	T	17	13	10	5	0

Column Reduction

		L_1	L_2	L_3	L_4	L_5
Operator	P	0	3	5	4	0
	Q	5	6	7	6	0
	R	0	0	0	0	5
	S	5	7	7	5	0
	T	1	2	6	5	0

∴ The minimum number of lines to cover all zeros is 3 which is less than 5, the above matrix will not give optimal solution.

∴ Subtract the least uncovered element which is 1 from all uncovered elements and add it to all the elements lying at the intersection of two lines, we get the following matrix :

		L_1	L_2	L_3	L_4	L_5
Operator	P	0	3	5	4	1
	Q	4	5	6	5	0
	R	0	0	0	0	6
	S	4	6	6	4	0
	T	0	1	5	4	0

∴ The minimum lines drawn to cover all zero are 3. Repeating the above procedure, the matrix is

NOTES

		L_1	L_2	L_3	L_4	L_5
Operator	P	0	2	4	3	1
	Q	4	4	5	4	0
	R	1	0	0	0	7
	S	4	5	5	3	0
	T	0	0	4	3	0

The minimum number of lines to cover all zeros is 4 which is less than 5 the resultant matrix is

		L_1	L_2	L_3	L_4	L_5
Operator	P	0	2	4	3	4
	Q	1	1	2	1	0
	R	1	0	0	0	10
	S	1	2	2	0	0
	T	0	0	4	3	3

∴ The minimum number of lines to cover all zeros is 5.

Hence the above matrix will give optimal solution.

		L_1	L_2	L_3	L_4	L_5
Operator	P	[0]	2	4	3	4
	Q	1	1	2	1	[0]
	R	1	~~0~~	[0]	~~0~~	10
	S	1	2	2	[0]	~~0~~
	T	~~0~~	[0]	4	3	3

And the assignment of operator setting lathe time is given by

$P \rightarrow L_1 = 20$

$Q \rightarrow L_2 = 40$

$R \rightarrow L_3 = 35$

$S \rightarrow L_4 = 37$

$L \rightarrow L_5 = 28$

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The maximum profit per week $25 \times 160 = 4000$.

Example 7.6. *A small school has five teachers teaching five different subjects. All the five teachers are capable of teaching all the five subjects. The output per day of the teacher and course coverage (%) for each subject are given below.*

NOTES

Teachers	1	2	3	4	5
A	7	9	4	8	6
B	4	9	5	7	8
C	8	5	7	9	8
D	6	5	8	10	10
E	7	8	10	9	9
(Course Coverage %)	2	3	2	3	4

If teacher D is not available, will your answer be different ?

Solution. We can multiply the output of teachers for different subjects with coverage of course (%), the resultant marix is shown as :

Teachers		1	2	3	4	5
	A	14	27	8	24	24
	B	8	27	10	21	32
	C	16	15	14	27	32
	D	12	15	16	30	40
	E	14	24	20	27	36

∵ The problem is of maximization.

∴ We will deduct the maximum element, i.e., 40. The resultant loss matrix will be

Teachers	Subjects	1	2	3	4	5
	A	26	13	32	16	16
	B	32	13	30	19	8
	C	24	25	26	13	8
	D	28	25	24	10	0
	E	26	16	20	13	4

Row Reduction

	1	2	3	4	5
A	13	0	19	3	3
B	24	5	22	11	0
C	16	17	18	5	0
D	28	25	24	10	0
E	22	12	16	9	0

Column Reduction

NOTES

Now subtracting the main elements of each column firm all its elements.

	1	2	3	4	5
A	0	0	3	0	3
B	11	5	6	8	0
C	3	17	2	2	0
D	15	25	8	7	0
E	9	12	0	6	0

∴ There are 3 lines, this is not the optimal solution.

So, we will follow the deduction method.

	1	2	3	4	5
A	0	0	3	0	5
B	9	3	4	6	0
C	1	15	0	0	0
D	13	23	6	5	0
E	9	12	0	6	2

Still the number of lines are less.

∴ Again the same procedure will be followed.

	1	2	3	4	5
A	0	0	3	0	8
B	6	0	1	3	0
C	1	15	0	0	3
D	10	20	3	2	0
E	9	12	0	6	5

Optimum solution is obtained.

	1	2	3	4	5
A	[0]	0	3	~~0~~	8
B	6	[0]	1	3	~~0~~
C	1	15	~~0~~	[0]	3
D	10	20	3	2	[0]
E	9	12	[0]	6	3

$A \rightarrow 1 \quad = 14$

$B \rightarrow 2 \quad = 27$
$C \rightarrow 4 \quad = 27$
$D \rightarrow 5 \quad = 40$
$E \rightarrow 3 \quad = 20$

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7.5 SUMMARY

- In real life situations, problems arise where a number of resources have to be allotted to a number of activities. In a sense, a special case of the transportation model is the Assignment Model. This model is used when the resources, have to be assigned to the tasks, i.e., assign *n* persons to *n* different type of jobs. Since different types of resources whether human, i.e., men or material, machines, etc., have different efficiency of performing different types of jobs and it involves different costs, the problem is how to assign such resources to jobs so that total cost is minimized or given objective is optimized.
- An assignment problem is, in fact, a completely degenerate form of a transportation problem.
- **Complete Enumeration Method**.
  In this method costs for all possible assignments are worked out and the one having the minimum cost is termed as the optimal solution. This method, for obvious reasons, can only be used for small problems.
- **Simplex Method**
  In this method the simplex algorithm is used and we

  Minimize or Maximize $Z = \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} \; x_{ij}$
- **Transportation Method**
  We have earlier mentioned that assignment model is a special case of transportation model, so it should be possible to solve it by transportation method. However, we know that optimality test in the transportation method requires that there should be $n + n - 1 = 2n - 1$ basic variables, the solution obtained by this method would be severally degenerate.
- **Hungarian Assignment Method or HAM (Minimization case)**
  This method was developed by Hungarian mathematician D Koning and is also known as the *Flood's Technique* or the *reduced matrix method*. It is a simpler and more efficient method of solving the assignment problems.
  One of the major applications of the assignment models is in the travelling salesman problem.

7.6 REVIEW AND DISCUSSION QUESTIONS

1. (*a*) Show that assignment model is a special case of transportation model.
   (*b*) Consider the problem of assigning five operators to five machines. The assignment costs are as follows.

NOTES

		Operators				
		I	II	III	IV	V
Machines	A	10	5	13	15	16
	B	3	9	18	3	6
	C	10	7	2	2	2
	D	5	11	7	7	12
	E	7	9	4	4	12

Assign the operators to different machines so that total cost is minimized.

2. Six machines M_1, M_2, M_3, M_4, M_5 and M_6 are to be located in six places P_1, P_2, P_3, P_4, P_5 and P_6. C_{ij} the cost of locating machine Mi at place Pj is given in he following matrix.

	P_1	P_2	P_3	P_4	P_5	P_6
M_1	20	23	18	10	16	20
M_2	50	20	17	16	15	11
M_3	60	30	40	55	8	7
M_4	6	7	10	20	25	9
M_5	18	19	28	17	60	70
M_6	9	10	20	30	40	55

Formulate an LP model to determine an optimal assignment. Write the objective function and the constraints in detail. Define any symbol used. Find an optimal layout by assignment techniques of linear programming.

3. (*a*) Discuss assignment model. Indicate a method of solving an assignment problem.

(*b*) A Company is faced with the problem of assigning six different machines to five different jobs. The costs estimated in hundreds of rupees are give in the table below.

	1	2	3	4	5
1	2·5	5	1	6	2
2	2	5	1·5	7	3
3	3	6·5	2	8	3
4	3·5	7	2	9	4·5
5	4	7	3	9	6
6	6	9	5	10	6

Solve the problem assuming that the objective is to minimize the total cost.

4. Five new machines are to be located in machine shop. There are five possible locations in which machines can be located. C_{ij} the cost of placing machine i in place j is give in the table below.

NOTES

Jobs

Machine		1	2	3	4	5
	1	15	10	25	25	10
	2	1	8	10	20	2
	3	8	9	17	20	10
	4	14	10	25	27	15
	5	10	8	25	27	12

It is required to place the machines at suitable places so as to minimize the total cost.

(*i*) Formulate an LP model to find an optimal assignment.

(*ii*) Solve the problem by assignment technique of LP.

5. Solve the following assignment problem :

	I	II	III	IV	V
1	11	17	8	16	20
2	9	7	12	6	15
3	13	16	15	12	16
4	21	24	17	28	26
5	14	10	12	11	15

6. A team of 5 horses and 5 riders has entered a jumping show contest. The number of penalty points to be expected when each rider rides any horse is shown below.

Horse		R_1	R_2	R_3	R_4	R_5
	H_1	5	3	4	7	2
	H_2	2	3	7	6	5
	H_3	4	1	5	2	4
	H_4	6	8	1	2	3
	H_5	4	2	5	7	1

How should the horses be allotted to the riders so as to minimise the expected loss of the team?

7. A Company has five jobs to be done. The following matrix shows the return in rupees of assigning *i*th machine (i = 1, 2, ……, 5) to the job (j = 1, 2, ……, 5). Assign the five jobs to the five machines so as to maximize the total expected profit.

Job

NOTES

Machines		1	2	3	4	5
	1	5	11	10	12	4
	2	2	4	6	3	5
	3	3	12	5	14	6
	4	6	14	4	11	7
	5	7	9	8	12	5

8. The owner of a small machine shop has four machinists available to assign to jobs for the day. Five jobs are offered with the expected profit in rupees for each machinist on each job being as follows. Find the assignment of machinists to jobs that will return in a maximum profit. Which job should be declined ?

Job

UNIT 8: THEORY OF GAMES

NOTES

Structure

8.1 INTRODUCTION

Till now we have used criteria for decision under uncertainty assuming that 'state of nature' is the opponent. Here the 'nature' is in the form of many possible outcomes of some action. When there are many possible outcomes or state of nature, one can't predict what will happen, it can only be predicted as a probability of a happening or occurrence. Such state of nature is beyond the control of any organization. Here certain happenings (or state of nature) like shift in the life style of people effecting demand; higher and better technology products made available in future at cheaper rates etc. effect-the pay-off and will decide the action of the decision-maker.

However, in real life situations, there are many competitors of any organization. Competition is the essence of existence of individuals and organizations. In modern day life no monopolistic situations exist in free economy. There are always two or more than two individuals or organizations making decisions and each wants the outcome in their favour as there is a conflict in the interest of each party. When decisions have to be made under conditions of uncertainty, two or more 'intelligent' opponents are involved and each one wants to optimize decision in his favour at the cost of other, Game Theory approach is involved. When we talk of 'intelligent' opponents what we mean is that in competitive situations each participant acts in a rational manner and does his best to resolve the situation in his favour.

This approach was developed by Professor John Von Newman and Oscar Morgensten when they published a book,. 'The theory of Games and Economic Behaviour' Games Theory is now widely used in economics, business and administration and many humanity disciplines as also by armed forces for training purposes. It is a useful scientific approach to rational decision-making.

8.2 TERMS USED IN GAME THEORY

Following are some important terms used in Game Theory:

1. **Player:** An opponent is referred to as a player.

NOTES

2. **Strategies:** Each player has a number of choices, these are called the strategies.
3. **Outcomes or Payoff:** Outcome of a game when different alternatives are adopted by the competing players, the gains or losses are called the payoffs.
4. **Two persons zero-sum game:** When only two players are involved in the game and the gains made by one player are equal to the loss of the other, it is called two persons zero-sum game. This may be the case when there are just two players in the game, i.e., assuming that there are only two types of beverages, tea and coffee. Any market share gained by the tea will equal the loss of market share of coffee. Since sum of the gains and losses is zero, this situation is called two persons zero sum gain.
5. ***n*, persons game:** A game in which n persons are involved is called n persons game. When n is more than two, i.e., more than two persons are involved the games become complex and difficult to solve.
6. **Payoff matrix:** When the gains and losses, which result from different actions of the competitors, are represented in the form of a matrix in a table, it is called payoff matrix, we have already seen many payoff matrix tables in earlier chapters.
7. **Decision of a game:** In game theory the decision criterion of optimality is adopted, i.e., a player, which wants to maximize his outcome, maximin, is used and the one who wishes to minimize his outcome, minimax is used.

A strategy basically relates to selection and use of one course of action out of various courses available to a player at a particular point of time. There are two types of strategies.

(*a*) *Pure strategy.* It is the course of action, which the player decides in advance. If there are 4 courses of action and the players select the third, then it is the third strategy which the player is using.

(*b*) *Mixed strategy.* In mixed strategy, the player decides his course of action by relating a fixed probability distribution. Some probability is associated with each course of action and decision to select one is done based on these probabilities.

8.3 LIMITATIONS OF GAME THEORY

Following are some limitations of Game Theory:

1. **Risk and uncertainty are not taken into account:** Since in pure strategies of game theory, no probability is associated with various courses of action, risk and uncertainty are not taken into account.
2. **A fixed number of competitors:** The theory assumes that there are a fixed number of competitors. In real life situations, there can be more than the expected number of players.
3. **Infinite courses of action:** Games Theory assumes finite number of courses of action available to each player. However, it is possible that a player may have infinite number of strategies available to him.
4. **Knowledge about strategies available to the opponent player:** The theory assumes that each player has knowledge about the strategies available to the other players. This may not always be the case.
5. **Zero sum game is not realistic:** The assumption that gains of one player are equal to the loss of the other player is not a realistic assumption.
6. **Knowledge of payoff in advance:** It is not always possible to know about the payoff

of a particular course of action.

7. **Rules of games do not permit tackling of all situations:** All games are played according a predetermined set of rules. These rules are based on certain assumptions and govern the behaviour pattern of the players. Many situations will fall outside the situation, which can be handled by these rules.

8.4 SITUATIONS OF TWO-PERSON ZERO-SUM PURE STRATEGY GAMES

As brought out earlier, the criterion used in Game theory is Maximin or Minimax.

Maximin Criterion. The player who is maximizing his outcome or payoff finds out his minimum gains from each strategy (course of action) and selects the maximum value out of these minimum gains.

Minimax Criterion. In this criterion the minimizing player determines the maximum loss from each strategy and then selects the strategy with minimum loss out of the maximum loss list.

Example 8.1. *Let us consider a two person zero sum game involving player A and player B. The strategies available to player A are A_1, A_2 and A_3 and to the player B are B_1 B_2. The payoff matrix is given blow by assuming the values.*

Player B

		B_1	B_2	Row Minima
Player A	A_1	12	4	4
	A_2	10	6	6
	A_3	8	9	8 Maximin
Column Maxima		12	9 Minimax	

Let us suppose that if A starts the game and selects A strategy, player B will select B_2 so that A gets minimum gains. Similarly, if A adopts A_3, B will adopt B_1 strategy to minimize the gain of A. Player A by selecting third strategy (A_3) is maximizing his minimum gain. Player A's selection is called *Maximum strategy*. Player B by selecting second strategy (B_2) is minimizing his maximum loss. Player B's selection is called *Minimax strategy*.

Example 8.2. *Consider the following payoff matrix which represents Player A's gain.*

Player B

		B_1	B_2	B_3	B_4	Row Minimum	
Player A	A_1	12	4	14	6	4	
	A_2	10	6	12	16	6	Maxi-min
	A_3	8	2	– 6	10	– 6	
Column Maximum		12	6	14	16		

NOTES

Minimax

When player A plays strategy A_1, he may gain 12, 4, 14 or 16 depending upon which strategy player B plays. But player A is guaranteed minimum of 4 in any case. In each row minimum gain guaranteed is 4, 6 and – 6. The maximum out of this is 6, so player A by selecting his second strategy A_2 is maximizing his minimum gains. Player B_1 he cannot loose more than maximum out of 12, 10 and 8, *i.e.,* whatever strategy A adopts, B cannot loose more than 12. Similarly, for strategies B_2, B_3 and B_4 the maximum looses are 12, 6, 14 and 16. Thus, by selecting B_2 he minimizes his maximum loss.

8.5 CONCEPT OF VALUE OF GAME

In game theory the value of game is important to both the players. For the maximizing player, it is the maximum guaranteed gain. For minimizing player, it is minimum loss. Consider the following game ith payoff matrix as shown :

Player B

		B_1	B_2	Row minimum
Player A	A_1	4	6	4 maximin
	A_2	– 8	3	– 8
Column Max		4	6	

Minimax

If player A adopts A_1 strategy, he gains 4, if player B adopts B_1 strategy he losses 4. In this case maxi (min) = min (max).

8.6 CONCEPT OF SADDLE POINT OR EQUILIBRIUM POINT

In a payoff matrix the value, which is the smallest in its row and the largest in the column, is called the *saddle point*.

Example 8.3. *Let us consider the following payoff nature to illustrate the concept of sddle or Equilibrium point :*

Player Y

		Y_1	Y_2
Player X	X_1	80	60
	X_2	100	120
	X_3	50	70

Saddle point can be found by :

1. Find out the minimum of the row and put a circle around it.
2. Find out the maximum of the column and put a square around it.
3. The value having bot the circle (□) and the square is the saddle point. It should be remembered that :
   (i) Saddle point may or may not exist in a game. It is not necessary that all payoff matrixes will have a saddle point.
   (ii) If there are more than one saddle points (which is a very rare occurrence), then the problem has more than one solution.
   (iii) The values of the game could be positive or negative.
   (iv) If the value of the game is zero, it is called a 'fair game'.

Example 8.4. *Find the ranges of value of P and Q, which will render the entry (2,) a saddle point for the game.*

Player B

	2	4	5
Player A	10	7	Q
	4	P	6

Solution. Let us determine the maximin and minimax in te payoff matrix provided above

Player B

		B_1	B_2	B_3	Row Minimum
Player A	A_1	2	4	5	
	A_2	10	7	Q	
	A_3	4	P	6	
	Column Max	10	7	6	

Maximum value = 7

Minimizing value 7 ignoring the values of P and Q now we want entry (2, 2), *i.e.*, A_2, B_2 to be the saddle point, *i.e.*, 7 should be the minimum in the row A_2, *i.e.,* Q should be more than 7, *i.e.*, $Q \geq 7$. Similarly, we want 7 to be highest number out of 4, 7 and P. It means that P should be equal to or lessor than 7, *i.e.*, $P \leq 7$.

Hence the range of P ≤ 7 and Q ≥ 7.

NOTES

Example 8.5. *Air Force of a country a wants to bomb the major enemy positions of country B. Bombers of country A have the option of attacking either high or low. If they fly low they can cause more damage to the enemy because of the accuracy they achieve. Country B will use its fighter aircrafts to intercept looking either high or low. If the bombers avoid the fighter they destroy 8 targets but if the fighter intercept them, no target can be destroyed. If the bombers are able to fly low, they can destroy 4 extra targets before being intercepted.*

Setup a game matrix. What advice you will give to commander of country A ?

Solution.

Country B

			FIGHTERS		
			HIGH	LOW	Row Min
Country A	BOMBER	HIGH	0	4	
		LOW	8	4	
	Column Max		8	4	

Let us find the saddle point. It can be easily seen that low-low entry 4 is the saddle point as it is minimum in its row and maximum in its column.

Value of the game V = 4

Bombers of country A and fighters of country B will both fly low entry (2 – 2)

Solution of two persons zero sum games with mixed strategies

In case of pure game, if a saddle point exists it straight away gives the optimal solution. Some games do not have saddle points. For example, let us consider the following zero-sum game :

Player B

		B_1	B_2	Row Minima
Player A	A_1	12	4	4
	A_2	10	6	6
	A_3	8	9	8 Maximin
Column Maxima		12	9 Minimax	

In this matrix, the minimax value 9 is greater than the maximin value 8. The game does not have a saddle point and the pure maximin-minimax strategies are not optimal. In this case the game is said to be unstable.

Because of the failure of minimax — maximin or pure strategies to give an optimal solution, we have to use mixed strategies. Each player plays all his strategies according to some probabilities rather than plays a pure strategy.

NOTES

Let $x_1, x_2, \ldots, x_m$ be the row probabilities by which player A selects his strategies.

Let $y_1, y_2, \ldots\ldots, y_n$ be the column probabilities of which player B selects his strategies then

$$\sum_{i=1}^{m} x_i = \sum_{j=1}^{n} y_j = 1$$

$$x_i, y_j \geq 0$$

Hence, if a_{ij} represents, the (i, j) entry of the game matrix, x_i and y_i will appaer as in the following matrix.

Player B

		y_1	y_2	y_3	....	y_n
Probabilities	x_1	a_{11}	a_{12}	a_{13}	....	a_{1n}
	x_2	a_{21}	a_{22}	a_{23}	....	a_{2n}
Player A	x_3	a_{31}	a_{32}	a_{33}	....	a_{3n}
	:	:	:	:		:
	:	:	:	:		:
	x_m	a_{m1}	a_{m2}	a_{m3}	....	a_{mn}

The solution of the mixed strategy problem is also based on the minimax criterion. However, if A selects x_i that maximizes the lowest expected payoff in a column and if B selects y_j, it minimizes the highest expected payoff in a row.

This will be illustrated in the examples that follow :

Odds Method

This method can be used only for 2 × 2-matrix games. In this method we ensure that sum of column odds and row odds is equal.

Finding out Odds

Step I. Take first row and find out the difference between values of cell (1, 1) and that of cell (1, 2) place this value in front of second row on the right side.

Step II. Take second row, find out the difference between the value of cell (2, 1) and that of cell (2, 2). Place it in front of the first row on the right side.

Step III. Take first column, find out the difference between the value of cell (1, 1) and that of value of cell (2, 1). Place it below the second column.

Step IV. Take second column, find out the difference between the value of cell (1, 2) and that of the value of cell (2, 2). Place this value below the first column.

Example 8.6. *Consider a modified form of "matching biased coins" game problem. The matching player is paid Rs. 8 if two coins turn both heads and Rs. 1 if the coins turn both tail. The non-matching player is paid Rs. 3 when the two coins do not match. Given the choice of being* *the matching or non-matching players, which one would you choose and what would be your strategy ?*

NOTES

Solution. Let us prepare the payoff matrix.

Player B

Player A		H	T
	H	8	–3
	T	–3	1

Let us see if the saddle point exists. Minimum of row one is – 3 and similarly minimum of row two is also – 3, a circle has been put around these figures. Maximum of column is 8 and that of column 2 is 1. A square has been put around these two figures. There is no value, which is the lowest in its row and maximum of its column. Hence no saddle point exists.

So, both the player will use mixed strategy.

Use of Odds Method

Player B

		B_1	B_2	
		H	T	Odds
Player A	A_1	8	– 3	4
	A_2	– 3	1	11
	Odds	4	11	

(a) Take first row – difference between the cell A1 B1 and A1 B2
   8 – (– 3) = 8 + 3 = 11 place it in front of second row.

(b) Take second row – difference between the cell A2 B1 and A2 B2
   – 3 – 1 = – 4 (ignore sign)

(c) Take first column – 3 – 1 = – 4 (ignore sign)

(d) Take second column 8 – (– 3) = 11

Value of the game

For finding out the value of the game, following formula is used :

Player B

		B_1	B_2	Odds
Player A	A_1	a_1	a_2	$(b_1 - b_2)$
	A_2	b_1	b_2	$(a_1 - a_2)$
	Odds	$(a_2 - b_2)$	$(a_1 - b_1)$	

$$\text{Value V} = \frac{a_1(b_1 - b_2) + b_1(a_1 - a_2)}{(b_1 - b_2) + (a_1 - a_2)}$$

$$\text{Probability of A}_1 = \frac{b_1 - b_2}{(b_1 - b_2) + (a_1 - a_2)}, \quad A_2 = \frac{a_1 - a_2}{(b_1 - b_2) + (a_1 - a_2)}$$

$$\text{Probability of B}_1 = \frac{a_2 - b_2}{(a_2 - b_2) + (a_1 - b_1)}, \quad B_2 = \frac{a_1 - b_1}{(a_2 - b_2) + (a_1 - b_1)}$$

$$\text{Game value} = \frac{8 \times 4 - 3 \times 11}{4 + 11} = \frac{-1}{15}$$

$$\text{Probabilities of A}_1 = \frac{4}{15}, \quad A_2 = \frac{11}{15}$$

$$\text{Probabilities of B}^1 = \frac{4}{15}, \quad B_2 = \frac{11}{15}$$

8.7 DOMINANCE METHOD OR PRINCIPLE OF DOMINANCE

This method basically states that if a particular strategy of a player dominates in values his other strategies then this strategy, which dominates, can be retained and what is dominated is deleted.

The dominance rule for column

Every value in the dominating column (s) must be equal to or less than the corresponding value of the dominated column.

The dominance rule for row

Every value in the dominating row (s) must be greater than or equal to the corresponding value of the dominated row. A given strategy can be dominated if on average its value is lesser than the average of two or more pure strategies. To illustrate this point, consider the following game:

B

		B_1	B_2	B_3
	A_1	8	3	4
A	A_2	2	9	8
	A_3	3	4	5

This game has no saddle point. Also none of the pure strategies of A (A_1, A_2, A_3) is lesser in value to any of his other pure strategies. Let us find out the average of A's pure and econd pure strategies (A_1 and A_2).

$$\left(\frac{(8+2)}{2}, \frac{(3+9)}{2}, \frac{(4+8)}{2}\right) = (5, 6, 6)$$

This is superior to A's third pure strategy (A_3). Hence strategy A_3 my be deleted and the matrix becomes

B

B_1	B_2	B_3

A	A_1	8	3	4
	A_2	2	9	8

NOTES

Sometimes game, which is reduced by dominance method, shows a saddle point but in original matrix there was no saddle point. This saddle point must be ignored since it does not have the properties of a saddle point, *i.e.,* least value in its row and the highest value in its column.

Example 8.7. *Reduce the following game bydominance and find the game value :*

Player B

		I	II	III	IV
Player A	I	*3*	*2*	*4*	*0*
	II	3	4	2	4
	III	4	2	4	0
	IV	0	4	0	8

Solution. Let us find if there is a saddle point in the matrix. This matrix has no saddle point. From player A's point of view, row III dominates row I as every value of row IV is either equal to or greater than every value in row I. Hence, row I can be deleted. The reduced matrix is

Player B

		I	II	III	IV
Player A	II	3	4	2	4
	III	4	2	4	0
	IV	0	4	0	8

From player B's point of view, column III dominates column I as every value of column III is equal to or lesser than the value of column I. Hence, column I can be deleted. The resulting matrix is

Player B

		II	III	IV
Player A	II	4	2	4
	III	2	4	0
	IV	4	0	8

In the above matrix, no single row or column dominates another row or column. Let us find if average of any two rows dominates the pure strategy of the other. There is no such possibility. Now let us try if the average of any two columns dominates the third, *i.e.,* if the average value of the two columns is equal to or less than the third average of columns III and IV is $\frac{(2+4)}{2}, \frac{(40+2)}{2}, \frac{(0+8)}{2}$ = (3, 2, 4). This value is equal to or lesser than value of column II, so column II can be deleted. The resulting matrix is

Player B

III IV

	II	2	4
Player A	III	4	0
	IV	0	8

Now, row II is dominated by average of row III and IV $\frac{(4+0)}{2}, \frac{(0+8)}{2} = (2, 4)$. Hence row II is deleted. This result in the reduced matrix shown below.

		Player B	
		III	IV
Player A	III	4	0
	IV	0	8

This is now a 2×2 matrix and can be solved by Odds method.

Step I. Subtract the two digits of column III and write them under column IV (ignoring signs).

Step II. Subtract the two digits of column III and write them under column IV (ignoring signs).

Step III. Subtract two digits of row III and write in front of row II (ignoring signs).

Step IV. Subtract two digits of row II and write it in front of row III (ignoring sign)

The resulting matrix with odds is as follows :

Probability of player A III 8/12, IV 4/12, *i.e.*, 2/3, 1/3

Probability of player B III 8/12, I 4/12,*i.e.*, 2/3, 1/3

$$\text{Value of the game } = \frac{a_1(b_1 - b_2) + b_1(a_1 - a_2)}{(b_1 - b_2) + (a_1 - a_2)} = \frac{4 \times 8 + 0 \times 4}{8 + 4} = \frac{32}{12} = \frac{8}{3}$$

Sub-Games Method for $2 \times n$ or $m \times 2$ Games

In this method we sub-divide the given game ($2 \times n$ or $m \times 2$) into a number of 2×2 games. Now, each of these 2×2 games can be solved and then optimal strategies are selected. These are games when one of the players has 2 alternatives; where as the other player has more than two alternatives. When there is no saddle point or the game cannot be solved by using the dominance method the sub-games method is very useful. It is suitable when the number of alternatives is limited to 4. In case of large number of alternatives, the solution becomes lengthy and complicated. It follows the following procedure :

Step I. Divide the $2 \times n$ or $m \times 2$ game matrix in 2×2 matrix sub-games.

Step II. Take up each game one by one and find out if a saddle point exists. Such a sub-game has pure strategies.

Step III. If the sub-game has no saddle point, then use odds method to solve the sub-game.

Step IV. Select the best sub-game out of all the sub-games from the point of view of the player who has more than two alternatives.

Step V. Find out the strategies of this selected sub-game. This is applicable to both the players and for the entire game.

Step VI. Find out the value of the selected sub-game, this will be the value of the whole game.

NOTES

Example 11.8. *Two airlines A and B operate their flights to an island and are interested in increasing their market share. Airline A has two alternatives, it either advertises its special fare or it advertises its features unique to it. On the other hand, airlines B have three alternatives of doing nothing, advertising their special fares or advertising their own special features. The matrix showing gains and losses of the two airlines in lakhs of rupees is shown below. Positive values favour airline or A and negative values favour airline B. Find the value of the game and best strategy by both the airlines using sub-game method.*

Solution.

Airline B

Airline A		B_1	B_2	B_3
	A_1	350	– 100	– 75
	A_2	200	180	175

A_1 – Advertising special fares
A_2 – Advertise special features
B_1 – Do nothing
B_2 – Advertise special fares
B_3 – Advertise special features

This game has to be solved by sub-games method as per the requirement of the question.

Step I. Let us divide the complete game (2 3) game as (× 2) game

Sub game I

B

A		B_1	B_2
	A_1	350	– 100
	A_2	200	180

Sub-game II

B

A		B_1	B_2
	A_1	35	– 75
	A_2	200	175

Sub-game III

B

		B1	B2
A	A1	– 100	– 75
	A2	180	175

Step II. Solve all thee sub-games

Solution to sub-game I

		B1	B2	Odds
A	A1	350	– 100	50
	A2	200	180	450
	Odds	250	150	

It has a saddle point, as minimum of row A_2 is also the maximum of column B_2.

Value o game = 180

Solution to sub-game II

		B1	B2
A	A1	35	– 75
	A2	200	175

As minimum of row A_2 is the maximum of column B_3, A_2, B_3 is the saddle point.

Value of game = 175

Solution to sub-game III

		B1	B2
A	A1	– 100	– 75
	A2	200	175

Value of the game = 175

Step II. Select the best sub-game from the point of view of the player who has more alternatives, *i.e.,* B.

Sub-game **Value**

I	180
II	175
III	175

NOTES

B will select that game which has minimum V.

B will select either sub-game II or III as both have equal V.

Step IV. Now, we find out the probabilities for the player A and B using their strateies while using sub-game II or III.

Sub-game II

		B: B_1	B_3	Odds
A	A_1	35	– 75	25
	A_2	200	175	120
Odds		250	165	

Sub-game III

		B: B_2	B_3	Odds
A	A_1	– 100	– 75	5
	A_2	180	175	175
Odds		250	280	

Probability of A to selectProbability of A to elect

Strategy $A_1 = \frac{25}{415}$ Strategy $A_1 = \frac{5}{530}$

Strategy $A_2 = \frac{120}{415}$ Strategy $A_2 = \frac{175}{530}$

Probability of B to select *Probability of B to select*

Strategy $B_1 = \frac{250}{415}$ Strategy $B_1 = 0$

Strategy $B_2 = 0$ Strategy $B_2 = \frac{250}{530}$

Strategy $B_3 = \frac{165}{415}$ Strategy $B_3 = \frac{280}{530}$

Example 8.9. *Solve the following game b equal gains or probability method:*

		Player B: B_1	B_2
A	A_1	8	2
	A_2	4	6

Solution. Let p be the probability of player A selecting strategy A_1 so $(1 – p)$ is the probability of A selecting strategy A_2, Also, let q be the probability of player B selecting strategy B_1 then $(1 – q)$ will be the probability of B selecting strategy B_2. Redraw the matrix after introducing the probability.

Player B

B_1	B_2	Probability

Player A	A_1	6	10	p
	A_2	8	4	$(1-p)$
Probability		q	$(1-q)$	

If player B selects strategy B_1 then payoff to A will be $8p + (1 - p)$.

If player B selects strategy B_2 payoff to player A will be $2p + 6(1 - p)$.

Since payoff under both the situations must be equal.

$$8p + 4(1-p) = 2p + 6(1-p)$$
$$8p + 4 - 4p = 2p + 6 - 6p$$
$$8p = 2$$
$$P = \frac{1}{4}, \text{ and } (1-p) = \frac{3}{4}$$

Similarly, we can work out the pay off to player B

$$8q + 2(1-q) = 4q + 6(1-q)$$
$$8p + 2 - 2q = 4 + 6 - 6q$$
$$8q = 4,\ q = \frac{1}{2},\ (1-q) = \frac{1}{2}$$

Value of game = (Expected pay off of player A when player B uses strategy B_1) ×

(Probability of player B using strategy B_1) + (Expected payoff player A when player B uses strategy B_2) × (Probability of player B using strategy B_2)

$$= [\{8p + 4(1-p)\} + q + \{2p + 6(1-p)\} \times (1-q)]$$
$$= [(8p + 4 - 4p)\ q + (2p + 6 - 6p)(1-q)]$$
$$= [(4p + 4)\ q + (-4p + 6)(1-q)]$$
$$= [4pq + 4q + (6 - 6q - 4p + 4pq)]$$
$$= (4pq + 4q + 6 - 6q - 4p + 4pq)$$
$$= -4p - 2q + 8pq + 6$$

Substituting the value of p and q, we get the value of game.

$$\text{Value of game} = -4 \times \frac{1}{4} - 2 \times \frac{1}{2} + 8 \times \frac{1}{4} \times \frac{1}{2} + 6$$
$$= -1 - 1 + 1 + 6 = 5$$

Probability of player a selecting strategy

$$A_1 = \frac{1}{4},\quad A_2 = \frac{3}{4}$$

Probability if player B selecting strategy

$$B_1 = \frac{1}{2},\quad B_2 = \frac{1}{2}.$$

Example 11.10. *Player X is paid Rs. 10 if two coins turn both Heads and Rs. 2 if both coins turn both Tails. Player Y is paid Rs. 4 when the two coins do not match. If you had the choice of becoming player X or player Y, which one would you like to be and what will be your strategy ? Solve the problem using equal gains or probability method.*

Solution. Let us construct the payoff matrix for the given problem.

NOTES

		Player Y	
		Y_1	Y_2
Player X	X_1	10	– 4
	X_2	– 4	2

Let p be the probability of player X selecting strategy X_1 so, $(1 - p)$ is the probability of player X selecting strategy X_2, similarly, let q be the probability of player Y selecting strategy Y_1 then $(1 - q)$ will be the probability of player Y selecting strategy Y_2. If player Y selects strategy Y_1 then payoff to player X is

$$= 10\,p - 4\,(1 - p)$$

If player Y selects strategy Y_2 then payoff to player X is $-4p + 2\,(1 - p)$.

Since payoff in both situations must be equal, *i.e.*,

$$10p - 4 + 4p = -4p + 2 - 2p$$

$$10p + 4p + 4p + 2p = 2 + 4$$

$$20p = 6$$

$$p = \frac{6}{20} = \frac{3}{10}$$

$$(1 - p) = \frac{7}{10}$$

If player X selects strategy X_1 then payoff to player Y is

$10q - 4\,(1 - q)$

If player X selects strategy X_2 then payoff to player Y is

$-4q + 2\,(1 - q)$

These two pay-offs must be equal, *i.e.,*

$$10q + 4q = -4q + 2 - 2q$$

$$10q + 4q + 4q + 2q = 2 + 4$$

$$q = \frac{6}{20} = \frac{3}{10}$$

$$(1 - q) = \frac{7}{10}$$

Now let us calculate the value of the game.

V = (Expected pay off of player X when player Y uses strategy Y_1) × probability of player Y using strategy Y_1) + (Expected pay off of player Y using strategy Y_2) × (probability of player Y using strategy Y_2)

$$V = [(10p - 4\,(1 - p)\}\,q + \{-4p + 2\,(1 - p)\} \times (1 - q)]$$

$$= [(10p - 4 + 4p)\,q + (-4p + 2 - 2p)\,(1 - q)]$$

$$= 10pq - 4q + 4pq + (2 - 6p)\,(1 - q)$$

$$= 14pq - 4q + 2 - 2q - 6p + 6pq$$

$$= 20pq - 6p - 6q + 2$$

Substituting the value of p and q.

$$P = q = \frac{3}{10}$$

$$= 20 \times \frac{3}{10} \times \frac{3}{10} - \frac{(6 \times 3)}{10} - \frac{(6 \times 3)}{10} + 2$$

$$= 1.8 - 1.8 - 1.8 + 2$$

$$= 0.2$$

Pay off Player X Uses X_1

$$10p - 4\ (1 - q) = 10 \times \frac{3}{10} - 4\left(1 - \frac{3}{10}\right)$$

$$= 3 - (4 \times 7)\ /10$$

$$= 3 - 28/10 = 1/5$$

Player X use X_2

$$4p + 2\ (1 - p) = -4 \times \frac{3}{10} + 2\left(1 - \frac{3}{10}\right)$$

$$= \frac{-12}{10} + \frac{14}{10}$$

$$= \frac{1}{5}$$

Pay off player Y uses $Y_1 = 10q - 4\ (1 - q) = \frac{1}{5}$

$$Y_2 = \frac{1}{5}$$

So, whether I am player X or Y, I have equal gains and equal probabilities.

Graphical Solution of (2 × *n*) and (*m* × 2) Games

Such solutions are possible only to the games in which at least one of the players has only two strategies. Let us consider a (2 × *n*) game of player A and B. Player A has only two strategies with probabilities p_1 and p_2 where $p_2 = 1 - p$ and player B has *n* strategies. Te game does not have a saddle point.

	Probability	Player B				
		B_1	B_2	B_3	…………	B_n
Player A	p_1	a_{11}	a_{12}	a_{13}	…………	a_{1n}
$p_2 = 1 - p_1$	p_2	a_{21}	a_{22}	a_{23}	…………	a_{2n}

A’s expected payoff corresponding to the pure strategies of B are as follows :

B’s pure strategy A’s expected payoff

B_1 $a11\ p_1 + a21\ (1 - p_1) = (a11 - a21)\ p_1 + a21$

B_2 $a12\ p_1 + a22\ (1 - p_1) = (a12 - a22)\ p_1 + a22$

B_3 $a13\ p_1 + a23\ (1 - p_1) = (a13 - a23)\ p_1 + a23$

: : : : :

B_n $a_1 n\ p_1 + a_2 n\ (1 - p_1) = (a_1 n - a_2 n)\ p_1 + a_2 n$

A should selects such a value of p_1 in such a manner that it maximizes his minimum payoff.

This can be done by plotting the payoff equations as straight line of functions of p_1.

The steps involved in this solutions are as follows :

Step I. The game must be reduced to such a sub-game that at least one of the players has only two strategies.

NOTES

Step II. Take the probability of two alternatives of a player (say A) having only two strategies as p_1 and $(1 - p_1)$. We formulate equations of net gain of A from different strategies of B.

Step III. Two parallel lines are drawn on the graph to include the boundaries of two strategies of first player say A.

Step IV. Pay off equations as functions of probabilities of two alternatives of A for different strategies of player B are plotted on the graph as straight-line functions of P.

Step V. If player A is maximizing, the point is identified where minimum expected gain is maximized, on the other hand, in case of minimizing player B, the point as identified where maximum loss is minimized.

The method will be demonstrated with the help of an example.

Example 11.11. *Solve the following game using the graphic method.*

				B	
		1	*2*	*3*	*4*
A	1	2	2	3	– 1
	2	4	3	2	6

Solution. The game does not have a saddle point. A's expected payoff according to the pure strategies of B is shown in the matrix below. p is the probability of A selecting strategy 1 and $(1 - p_1)$ is the probability of A selecting strategy 2.

B's Strategy	A's expected payoff
B_1	$2p + 4(1 - p) = -2p + 4$
B_2	$2p + 3(1 - p) = -p + 3$
B_3	$3p + 2(1 - p) = p + 2$
B_4	$-p + 6(1 - p) = -7p + 6$

NOTES

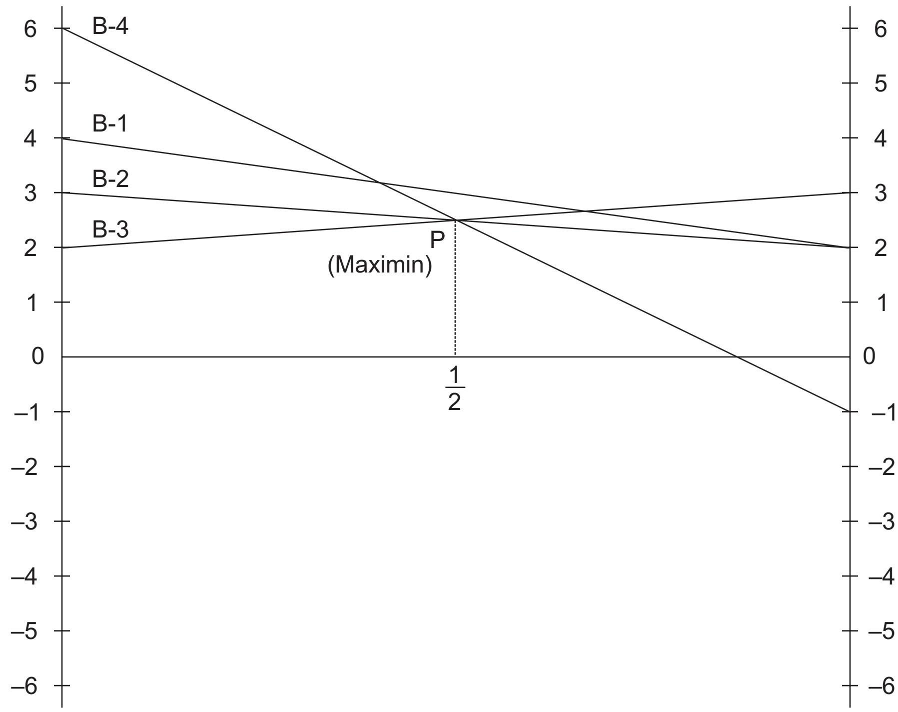

Fig. 8.1

Let us plot $-2p + 4$ when $p = 0$ value = 4

when $p = 1$ value = 2

$-p + 3$ when $p = 0$ value = 3

when $p = 1$ value = 2

$p + 2$ when $p = 0$ value = 2, $p = 1$, value = 3

$-7p + 6$ when $p = 0$ value = 6

$p = 1$ value = – 1

It can be seen from the graph that maximin occurs at $p = \frac{1}{2}$. This is the point of intersection of any two of the lines. Hence As optimal strategy is $p = \frac{1}{2}$, $1 - p = \frac{1}{2}$. The value of game can be found out by substituting the value of p in the equation of any of the lines passing through P.

$$V = \begin{Bmatrix} -\frac{1}{2} + 3 = \frac{5}{2} \\ \frac{1}{2} + 2 = \frac{5}{2} \\ \frac{-7}{2} + 6 = \frac{5}{2} \end{Bmatrix}$$

P is the point of intersection of any three of the lines B_2, B_3 and B_4. To find out the optimal strategies of B as three lines pass through P, it indicates that B can mix all the three strategies, *i.e.*, B_2, B_3 and B_4. The combination of $B_2 - B_3$, $B_3 - B_4$, $B_2 - B_4$ and must be considered.

Example 8.12. *A is paid Rs. 8 if coins turn both heads and Rs. 1 if two coins turn both tail B. wins Rs 3 when the two coins do not match give the choices t be a or B. Find the values of Game.*

NOTES

$$A \quad A \quad \begin{matrix} & B & \\ & H & T \\ H & \textcircled{8} & -3 \\ T & -3 & 1 \end{matrix}$$

Solution. The above problem does not have saddle point, the players will use mixed strategy. Let p_1 be the probability of player A selecting strategies I $(1 - p_1)$ is probability the player A will select strategy II

Similarly, q_1 is the probability. of player B selecting strategy II

$(1 - q_1)$ is probability of player B selecting I.

Then payoff of A is

$$= 8p_1 - 3\ (1 - p_1).$$

If B selects strategy II the payoff

$$= -\ 3p_1 + 1\ (1 - p_1).$$

$\because$ Gains are equal

$$8p_1 - 3\ (1 - p_1) = -\ 3p_1 + (1 - p_1)$$

$$8p_1 - 3 + 3p_1 = -\ 3p_1 + 1 - 1$$

$$11p_1 = -\ 41 + 1$$

$$15\ p_1 = 4$$

$$p_1 = .\frac{4}{15}$$

$$\therefore \qquad 1 - p_1 = \frac{11}{15} \quad 1 - \frac{4}{15}$$

Probability of B

$$8q_1 - 3\ (1 - q_1) = -3q_1 + 1\ (1 - q_1)$$

$$8q_1 - 3 + 3q_1 = -\ 3q_1 + 1 - q_1$$

$$11q_1 - 3 = \ 41 + 1$$

$$15q_1 = 4$$

$$q_1 = \frac{4}{15}$$

$$1 - q_1 = .\frac{11}{15}$$

$$\left(\begin{array}{ll} \text{V} = & \text{Expected payoff to player A} \\ \text{When} & \text{B uses strategy I} \\ & \text{X probability of player B selecting strategy I} \\ + & \text{Expected pay off is player A} \\ \text{When} & \text{B uses strategy II} \\ & \text{X probability of player B selecting strategy II} \end{array}\right)$$

$$V = [\{8p_1 - 3(1 - p_1\}q_1] + [\{(-3p_1) + 1\ (1 - p_1)\}\ (1 - q_1)]$$

Putting the value of $(p_1)\ (q_1)\ (1 - q_1)\ (1 - p_1)$

$$= 8\left(\frac{4}{5}\right) - 3\left(\frac{11}{5}\right)\left(\frac{4}{15}\right) + (-3)\left(\frac{4}{15}\right) + 1\left(\frac{11}{15}\right)\left(\frac{11}{15}\right)\left(-\frac{1}{15}\right)\frac{4}{15} + \frac{-1}{15}\left(\frac{11}{15}\right)$$

$$= \frac{-1}{15}$$

It is better to B as the value of Game is positive.

Example 11.13. **S***olve the following game.*

		B	
		I	*II*
A	*I*	*–6*	*7*
	II	*+4*	*–5*
	III	*–1*	*–2*
	IV	*–2*	*5*
	V	*7*	*–6*

Solution.

		B	
	I	–6	7
	II	4	–5
A	III	–1	–2
	IV	–2	5
	V	7	–6

The above game does not have a saddle point and no row or column is dominated.

∴ W will ply sub-game method

Sub-game I

		B		
		I	II	Odds
A	I	– 6	7	– 4 – (5) = 9
	II	4	–5	– 6 – 7 = 13
	Odds	7 – (–5) = 12	–6 – 4 = 10	

$$V = \frac{-6(9)+4(13)}{9+13} = \frac{-1}{11}$$

Sub-game II

		B		
		I	II	Odds
A	I	– 6	7	–1 – (–2) = 1
	II	–1	–2	– 6 – (7) = 13
		7 – (–2) = 9	– 6 – (–1) = 5	

$$V = \frac{(-6)(13)+7(13)}{13+13} = \frac{13}{26} = \frac{1}{2}$$

Sub-game III

NOTES

		B	
		I	II
A	I	– 6	7
	IV	2	5

At saddle point V = (–2)

Sub-Game IV

	B			
		I	II	Odds
A	I	– 6	7	7 – (– 6) = 13
	V	7	– 6	– 6 –7 = 13
	Odds	– 7 – (– 6) = 13	– 6 – 7 = 13	
		= 13	= 13	

$$V= \frac{(-6)\,(13)+7\,(13)}{13+13}=\frac{13}{26}=\frac{1}{2}$$

Sub-Game V

		B	
		I	II
A	II	4	–5
	III	–1	–2

At saddle point V = –2

Sub-Game VI

		B		
		I	II	
A	II	4	–5	–2–5 = 7
	IV	–2	5	–4 – (–5) = 9
		–5 – 5	4 (– 2)	
		= 10	= 6	

$$V = \frac{4\,(7)+(-2)\,(9)}{7+9}=\frac{10}{16}=\frac{5}{8}$$

Sub-game VII

B	
I	II

A	II	4	–5
	IV	7	– 6

t saddle point V = –5

NOTES

Sub-game VIII

		B I	B II	Odds
A	II	–1	–2	–2 –5 = 7
	IV	–2	5	–1 – (– 2) = 1
	Odds	– 2 – 5 = 7	1 – (–2) = 1	

$$V = \frac{(-1)(7) + (-2)(1)}{7+1}$$

$$= \frac{-9}{8}$$

Sub-game IX

		B I	B II
A	III	–1	–2
	V	7	–6

At saddle point

V = –21

Sub-game X

		B I	B II	Odds
A	IV	–2	5	7 – (– 6) = 13
	V	7	–6	–2 – 5 = 7
	Odds	5 – (– 6) = 11	–2 – 7 = 9	

$$V = .\frac{-2(13) + 7(7)}{13+7} = \frac{23}{20}$$

NOTES

Value of sub-games

I	II	III	IV	V	VI	VII	VIII	IX	X
$\frac{-1}{11}$	$-\frac{19}{11}$	–2	$\frac{1}{2}$	–2	$\frac{5}{8}$	–5	$-\frac{9}{8}$	–2	$\frac{23}{20}$

$\because$ The game having maximum V, *i.e.,* $\frac{23}{20}$. has been selected.

$$= \frac{23}{20}.$$

Optimum	I	II	III	IV	V
A	0	0	0	$\frac{13}{20}$	$\frac{7}{20}$
B	$\frac{11}{20}$	$\frac{9}{20}$	–	–	–

8.8 SUMMARY

- This approach was developed by Professor John Von Newman and Oscar Morgensten when they published a book,. 'The theory of Games and Economic Behaviour' Games Theory is now widely used in economics, business and administration and many humanity disciplines as also by armed forces for training purposes. It is a useful scientific approach to rational decision-making.
- **Player:** An opponent is referred to as a player.
- **Strategies:** Each player has a number of choices, these are called the strategies.
- **Outcomes or Payoff:** Outcome of a game when different alternatives are adopted by the competing players, the gains or losses are called the payoffs.
- **Two persons zero-sum game:** When only two players are involved in the game and the gains made by one player are equal to the loss of the other, it is called two persons zero-sum game.
- ***n*, persons game:** A game in which n persons are involved is called n persons game.
- **Decision of a game:** In game theory the decision criterion of optimality is adopted,
- **Pure strategy.** It is the course of action, which the player decides in advance.
- **Mixed strategy.** In mixed strategy, the player decides his course of action by relating a fixed probability distribution.
- This method basically states that if a particular strategy of a player dominates in values his other strategies then this strategy, which dominates, can be retained and what is dominated is deleted.

8.9 REVIEW AND DISCUSSION QUESTIONS

1. What do you understand by the term Theory of games ? Explain its use in decision-making.
2. How is decision-making related by Games theory ? What is the concept behind it ? Explain with suitable examples from business and industry.
3. What is a 'state of nature' ? Explain it taking suitable examples from real life situations.
4. What do you understand by payoff matrix ? How is it constructed ? Explain by taking examples.
5. What is a game in game theory ? What are the properties of a game ? Explain the "best strategy" on the basis of minimax criterion of optimality.
6. What assumptions are made in the theory of games ?
7. When is a competitive situation called a game ? What is the maximin criterion of optimality ?
8. (a) Give an example of the games theory as applicable to advertisement policies or strategies.

   (b) State three applications of game theory in marketing ?
9. Describe a two-person zero-sum game.
10. Let (a_{ij}) be the payoff matrix for a two-person zero sum game. If $\underline{v}$ denotes the maximin value and $\vec{v}$ the minimax value of the game, then show that $\vec{v} \geq \underline{v}$.
11. Explain the terms : Pure strategy, mixed strategy, saddle point, competitive games, payoff matrix, rectangular games.

    (a) Define the term 'strategy' and 'optimal strategy' with reference to Game theory.

    (*b*) Explain the Maximin and Minimax principle used in Game theory.
12. Define saddle point. Is it necessary that a game should always possess a saddle point ?
13. Explain "saddle point" and "dominance" as applied to a game. Illustrate with examples.
14. How do you solve a game when (*a*) Saddle point exists and (*b*) Saddle point does not exist ?
15. Show that for any zero-sum two-person game where there is no saddle point and for which λ's payoff matrix is

    $$= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

    the optimal strategies (x_1, x_2) and (y_1, y_2) for an B respectively are determined by

    $$\frac{x_1}{x_2} = \frac{a_{22} - a_{21}}{a_{11} - a_{12}}, \frac{y_1}{y_2} = \frac{a_{22} - a_{12}}{a_{11} - a_{21}}.$$

    What is the vale of the game to A ?
16. For the game $\begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$, where *a*, *b*, *c*, *d* are all non-negative, prove that he optimal strategies are :

    $$\left[\frac{c+d}{a+b+c+d}, \frac{a+b}{a+b+c+d}\right], \left[\frac{b+d}{a+b+c+d}, \frac{a+c}{a+b+c+d}\right] \text{and } v = \frac{ad-bc}{a+b+c+d}$$

UNIT 9: THE SEQUENCING PROBLEMS

NOTES

Structure

9.1 INTRODUCTION

A sequence is the order in which different jobs are to be performed. When there is a choice that a number of tasks can be performed in different orders, then the problem of sequencing arises. Such situations are very often encountered by manufacturing units, overhauling of equipments or aircraft engines, maintenance schedule of a large variety of equipment used in a factory, customers in a bank or car servicing garage and so on.

The basic concept behind sequencing is to use the available facilities in such a manner that the cost (and time) is minimized. The sequencing theory has been developed to solve difficult problems of using limited number of facilities in an optimal manner to get the best production and minimum costs.

Terms Commonly used

1. **Job:** These have to be sequenced, hence there should be a particular number of jobs (groups of tasks to be performed) say n to be processed.
2. **Machine:** Jobs have to be performed or processed on machines. It is a facility which has some processing capability.
3. **Loading:** Assigning of jobs to facilities and committing of facilities to jobs without specifying the time and sequence.
4. **Scheduling:** When the time and sequence of performing the job is specified, it is called *scheduling*.
5. **Sequencing:** Sequencing of operations refers to a systematic procedure of determining the order in which a series of jobs will be processed in a definite number, say k, facilities or machines.
6. **Processing Time:** Every operation that is required to be performed requires definite amount of time at each facility or machine when processing time is definite and certain, scheduling is easier as compared to the situation in which it is not known.
7. **Total Elapsed Time:** It is the time that lapses between the starting of first job and the completion of the last one.
8. **Idle Time:** The time for which the facilities or machine are not utilized during the total elapsed time.
9. **Technological Order:** It is the order which must be followed for completing a job. The requirement of the job dictates in which order various operations have to be performed, for example, painting cannot be done before welding.

10. **Passing not allowed:** If '*n*' jobs have to be processed through '*m*' machines in a particular order of M_1 , M_2 , M_3 then each job will go to machine M_1 first and then to M_2 and finally to M_3. This order cannot be passed.
11. **Static arrival Pattern:** If all the jobs to be done are received at the facilities simultaneously.
12. **Dynamic arrival Pattern:** Here the jobs keep arriving continuously.

Assumptions

In sequencing problems, the following assumptions are made :

(*i*) All machines can process only one job at a time.

(*ii*) No time is wasted in shifting a job from one machine to other.

(*iii*) Processing time of job on a machine has no relation with the order in which the job is processed.

(*iv*) All machines have different capability and capacity.

(*v*) All jobs are ready for processing.

(*vi*) Each job when put on the machine is completed.

(*vii*) All jobs are processed in specified order as soon as possible.

9.2 TYPES OF SEQUENCING PROBLEMS

The following types of sequencing problems will be discussed in this chapter:

(*a*) *n* jobs one machine case

(*b*) *n* jobs two machines case

(*c*) *n* jobs '*m*' machine case

(*d*) Two jobs '*m*' machines case.

The solution of these problems depends on many factors such as :

(*a*) The number of jobs to be scheduled

(*b*) The number of machines in the machine shop

(*c*) Type of manufacturing facility (slow shop or fast shop)

(*d*) Manner in which jobs arrive at the facility (static or dynamic)

(*e*) Criterion by which scheduling alternatives are to be evaluated.

As the number of jobs (*n*) and the number of machines (*m*) increases, the sequencing problems become more complex. In fact, no exact or optimal solutions exist for sequencing problems with large *n* and *m*. Simulation seems to be a better solution technique for real life scheduling problems.

n-Jobs One Machine Case

This case of a number of jobs to be processed on one facility is very common in real life situations. The number of cars to be serviced in a garage, number of engines to be overhauled in one workshop, number of patients to be treated by one doctor, number of different jobs to be machined on a lathe, etc, are the cases which can be solved by using the method under study. In all such cases we are all used to *'first come first served'* principle to give sense of satisfaction and justice to the waiting jobs. But if this is not the consideration, it is possible to get more favourable results in the interest of effectiveness and efficiency. The following assumptions are applicable :

(*a*) The job shop is static.

(*b*) Processing time of the job is known.

NOTES

The implication of the above assumption that job shop is static will mean that new job arrivals do not disturb the processing of *n* jobs already being processed and the new job **arrivals** wait to be attended to in next batch.

Shortest Processing Time (SPT) Rule

This rule says that jobs are sequenced in such a way that the job with least processing time is picked up first, followed by the job with the next Smallest Processing Time (SPT) and so on. This is referred to as *shortest processing time sequencing*. However, when the importance of the jobs to be performed varies, a different rule called Weight-Scheduling (Weight Scheduling Process Time) rule is used. Weights are allotted to jobs, greater weight meaning more important job. Let W_i be the weight allotted. By dividing the processing time by the weight factor, the tendency to move important job to an earlier position in the order is achieved.

$$\text{Weighted Mean low Time, (WMFT)} = \frac{\sum_{i=1}^{n} W_i f_i}{\sum_{i=1}^{n} W_i}$$

where f_i = flow time of job $i = W_i + t^i$

t_i = processing time of job i

WSPT rule for minimizing Weighted Mean-Flow Time (WMFT) puts *n* jobs in a sequence such that

$$\frac{t[1]}{W[1]} \leq \frac{t[2]}{W[2]} \leq \ldots \leq \frac{t[n]}{W[n]}$$

The numbers in brackets above define the position of the jobs in the optimal sequence.

Example 9.1. *Consider the 8 jobs with processing times, due dates and importance weights as shown below.*

8 jobs one machine case data

Task (*i*)	**Processing Time (t_i)**	**Due Date (d_i)**	**Importance Weight (W_i)**	$\frac{t_i}{W_i}$
1	5	15	1	5×0
2	8	10	2	4×0
3	6	15	3	2×0
4	3	25	1	3×0
5	10	20	2	5×0
6	14	40	3	4×7
7	7	45	2	3×5
8	3	50	1	3×0

From processing time t_i in the table the SPT sequence is 4–8–1–3–7–2–5–6 resulting in completion of these jobs at times 3, 6, 14, 20, 27, 36, 46, 60 respectively.

$$\text{WFT} = \frac{3+6+14+20+27+36+46+60}{8} = 26.5 \text{ hours}$$

NOTES

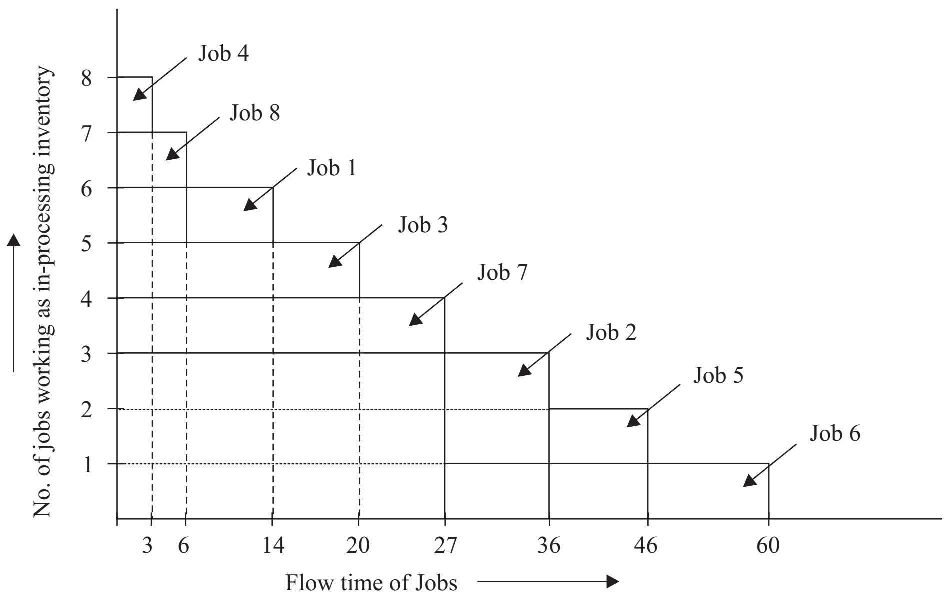

Fig. 9.1

The sequence is shown graphically above from which the number of tasks waiting as in-process inventory is seen to be 8 during 0–3, 7 during 3–6, 6 during 6–14, 5 during 14–20, 4 during 20–27, 3 during 27–36, 2 during 36–46 and one during 46–60. Thus, the average inventory is given by

$$\text{Average inventory} = \frac{6\times3+9\times1+12\times1+21\times3+28\times2+42\times3+52\times2+58\times1}{3+1+1+3+2+3+2+1}$$

$$= \; = 3.53 \text{ jobs.}$$

Weight Scheduling Process Time

If the important weights W_i were to be considered the WSPT could be used to minimize the Weighted Mean Flow Time (WMFT) to yield the sequence 3–4–8–2–7–6–5–1. This results by first choosing job with minimum $\frac{t_i}{W_i}$ in the table. The respective flow time of jobs in this sequence are 6, 9, 12, 21, 28, 42, 52, 58. Mean flow time is hours

$$\text{WMFT} = \frac{6\times3+9\times1+12\times1+21\times3+28\times2+42\times3+52\times2+58\times1}{3+1+1+3+2+3+2+1}$$

$$= \frac{18+9+12+63+56+126+104+58}{16} = \frac{446}{16} = 27.85 \text{ hours}$$

Example 9.2. *Eight jobs A, B, C, D, E, F, G and H arrive at one time to be processed on a single machine. Find out the optimal job sequence, when their operation time is give in the table below.*

Job (n)	Operation time in minutes
A	16
B	12
C	10
D	8

NOTES

E	7
F	4
G	2
H	1

Solution. For determining the optimal sequence, the jobs are selected in a non-descending operation time as follows :

Non-decreasing operation time sequence is H → G → F → E → D → C → B → A.

Total processing time

H = 1

G = 1 + 2 = 3

F = 1 + 2 + 4 = 7

E = 1 + 2 + 4 + 7 = 14

D = 1 + 2 + 4 + 7 + 8 = 22

C = 1 + 2 + 4 + 7 + 8 + 10 = 32

B = 1 + 2 + 4 + 7 + 8 + 10 + 12 = 44

A = 1 + 2 + 4 + 7 + 8 + 10 + 12 + 16 = 60

Average processing time = Total time/number of jobs = 183/8 = 23 minutes

In case the jobs are processed in the order of their arrival, *i.e.,* A → B → C → D → E → F → G → H the total processing time would have been as follows :

A = 16

B = 16 + 12 = 28

C = 16 + 12 + 10 = 38

D = 16 + 12 + 10 + 8 = 46

E = 16 + 12 + 10 + 8 + 7 = 53

F = 16 + 12 + 10 + 8 + 7 + 4 = 57

G = 16 + 12 + 10 + 8 + 7 + 4 + 2 = 59

H = 16 + 12 + 10 + 8 + 7 + 4 + 2 + 1 = 60

Average processing time = 357/8 = 44·6, which is much more than the previous time.

Priority Sequencing Rules

The following priority sequencing rules are generally followed in production/service system:

1. **First Come First Served(FCFS):** As explained earlier, it is followed to avoid any heart burns and avoidable controversies.
2. **Earliest Due Date (EDD):** In this rule, top priority is allotted to the waiting job, which has the earliest due/delivery date. In this case the order of arrival of the job and processing time it takes is ignored.
3. **Least Slack Rule (LS):** It gives top priority to the waiting job whose slack time is the least. Slack time is the difference between the length of time remaining until the job is due and the length of its operation time.

4. **Average Number of Jobs in the System:** It is defined as the average number of jobs remaining in the system (waiting or being processed) from the beginning of sequence through the time when the last job is finished.
5. **Average Job Lateness:** Jobs lateness is defined as the difference between the actual completion time of the job and its due date. Average job lateness is sum of lateness of all jobs divided by the number of jobs in the system. This is also called Average Job Tardiness.
6. **Average Earliness of Jobs:** If a job is completed before its due date, the lateness value is negative and the magnitude is referred as earliness of job. Mean earliness of the job is the sum of earliness of all jobs divided by the number of jobs in the system.
7. **Number of Tardy Jobs:** It is the number of jobs which are completed after the due date.

Sequencing n Jobs Through Two Machines

The sequencing algorithm for this case was developed by Johnson and is called *Johnson's Algorithm*. In this situation *n* jobs must be processed through machines M_1 and M_2. The processing time of all the *n* jobs on M_1 and M_2 is known and it is required to find the sequence, which minimizes the time to complete all the jobs.

Johnson's algorithm is based on the following assumptions:

(*i*) There are only two machines and the processing of all the jobs is done on both the machines in the same order, *i.e.*, first on M_1 and then on M_2 .

(*ii*) All jobs arrive at the same time (static arrival pattern) have no priority for job completion.

Johnson's algorithm involves following steps :

1. List operation time for each job on machine M_1 and M_2 .
2. Select the shortest operation or processing time in the above list.
3. If minimum-processing time is on M_1, place the corresponding job first in the sequence. If it is on M_2 , place the corresponding job last in the sequence. In case of tie in shortest processing time, it can be broken arbitrarily.
4. Eliminate the jobs which have already been sequenced as result of step 3.
5. Repeat steps 2 and 3 until all the jobs are sequenced.

Example 9.3. *Six jobs are to be sequenced, which require processing on two machines M_1 and M_2. The processing time in minutes for each of the six jobs on machines M_1 and M_2 is given below. All the jobs have to be processed in sequence M_1, M_2. Determine the optimum sequence for processing the jobs so that the total time of all the jobs is minimum. Use Johnson's algorithm.*

Jobs		1	2	3	4	5	6
Processing Time	Machine M_1	30	30	60	20	35	45
	Machine M_2	45	15	40	25	30	70

Solution.

Step I. Operation time or processing time for each jobs on M_1 and M_2 is provided in the question.

Step II. The shortest processing time is 15 for job 2 on M_2.

Step III. As the minimum processing time is on M_2, job 2 has to be kept last as follows:

					2

Step IV. We ignore job 2 and find out the shortest processing time of rest of jobs. Now the least processing time is 20 minutes on machine M_1 for job 4. Since it is on M_1, it is to be placed first as follows:

4					2

The next minimum processing time is 30 minutes for job 5 on M_2 and Job 1 on M_1. So, job 5 will be placed at the end. Job 1 will be sequenced earlier as shown below.

4	1			5	2

The next minimum processing time is 40 minutes for job 3 on M2, hence t is sequenced as follows :

4	1		3	5	2

Job 6 has to be sequenced in the gap or vacant space. The complete sequence in of the jobs is as follows.

4	1	6	3	5	2

The minimum time for six jobs on machine M_1 and M_2 can be shown with the help of a Gantt chart as shown below.

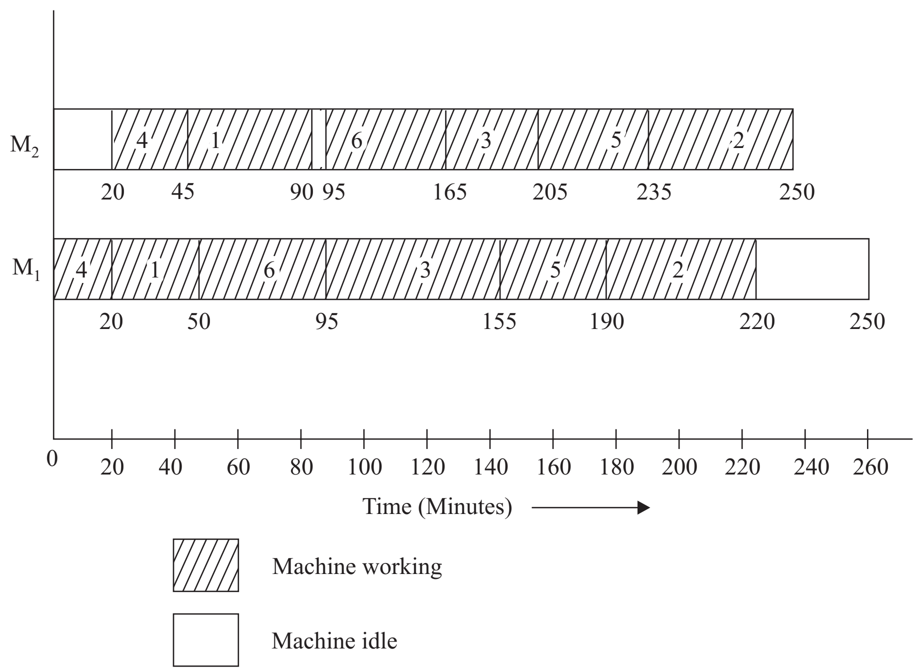

Fig. 9.2

The above figure shows idle time for M_1 (30 minutes) after the last job (2) has been processed. Idle time for M_2 is 20 minutes before job 4 is started and 5 minutes before processing 6 and finishing job 1. The percentage utilization of M1 = 250 – 30/250 = 88% and M2 = 250 – 25/250 = 90%.

Example 9.4. *A manufacturing company has 5 different jobs on two machines M_1 and M_2. The processing time for each of the jobs on M_1 and M_2 is given below. Decide the optimal sequence of processing of the jobs in order to minimize total time.*

Job. No.	Processing Time	
	M_1	M_2
1	8	6
2	12	7
3	5	11
4	3	9
5	6	14

Solution. The shortest processing time is 3 on M_1 for job 4 so it will be sequenced as follows:

4				

Next is job 3 with time 5 and M_1 , hece job 3 will be sequenced as

4	3			

Next minimum time is for jobs 1 on M this will be sequenced last.

4	3			1

After eliminating jobs 4, 3 and 1, the next with minimum time is job 5on M_1 so it will be placed as

4	3	5		1

Now, job 2 will be sequenced in the vacant space.

4	3	5	2	1

n Jobs 3 Machines Case

Johnson's algorithm which we have just applied can be extended and made use of in *n* jobs 3 machine case, if the following conditions hold good :

(*a*) Maximum processing time for a job on machine M_1 is greater than or equal to maximum processing time for the same job.

or

(*b*) Minimum processing time for a job on machine M_3 is greater than or equal to maximum processing time for a job on machine M_2.

The following assumptions are made :

(*a*) Every job is processed on all the three machines M_1, M_2 and M_3 in the same order, *i.e.,* the job is first processed on M_1 then on M_2 and then on M_3.

(*b*) The passing of jobs is not permitted.

(*c*) Processing time for each job on the machine M_1, M_2 and M_3 are known.

NOTES

In this procedure two dummy machines M_1' and M_2' are assumed in such a manner that the processing time of jobs on these machines can be calculated as

Processing time of jobs on M_1' = Processing time ($M_1 + M_2$)

Processing time of a job on M_2' = Processing time ($M_2 + M_3$)

After this Johnson's algorithm is applied on M_1' and M_2' to find out the optimal sequencing of jobs.

Example 9.5. *In a manufacturing process three operations have to be performed on machines M_1, M_2 and M_3 in order M_1, M_2 and M_3. Find out the optimum sequencing when the processing time for four jobs onthree machines is as follows:*

Job	M_1	M_2	M_3
1	3	8	13
2	12	6	14
3	5	4	9
4	2	6	12

Solution.

Step I. As the minimum processing time for job 2 on M_1 > maximum processing time for job 2 on M_2, Johnson's algorithm can be applied to this problem.

Step II. Let us combine the processing time of M_1 and M_2 and M_3 to form two dummy machines M_1' and M_2'. This is show matrix below.

Job	M'_1	M'_2
1	11 (3 + 8)	21 (8 + 13)
2	18 (12 + 6)	20 (6 + 14)
3	9 (5 + 4)	13 (4 + 9)
4	8 (2 + 6)	18 (6 + 12)

Step III. Apply Johnson's algorithm. Minimum time of 8 occurs for job 4 on M¢ hence it is sequenced first.

4	3	1	

The next minimum time is for job 3 on M¢1 so it is sequenced next to job 4. Next is job 1 and so on So the optimal sequencing is

4	3	1	2

Example 13.6. *Four Jobs 1, 2, 3 and 4 are to be processed on each of the five machines M_1, M_2, M_3, M_4 and M_5 in the order M_1, M_2, M_3, M_4, and M_5. Determine total minimum elapsed time if no passing off is allowed. Also find out the idle time of each of the machines. Processing time re given in the matrix below.*

NOTES

Job	Machines				
	M_1	M_2	M_3	M_4	M_5
1	8	4	6	3	9
2	7	6	4	5	10
3	6	5	3	2	8
4	9	2	1	4	6

Solution.

Step I. Find out if the condition minimum $e_i \geq$ maximum b_i, c_i and d_i is satisfied.

Job	Machines				
	M_1	M_2	M_3	M_4	M_5
1	8	4	6	3	9
2	7	6	4	5	10
3	6	5	3	2	8
4	9	2	1	4	6
	Minimum 6	Maximum 6	Maximum 6	Maximum 5	Minimum 6

This condition is satisfied hence we can convert the problem into four jobs and two fictitious machines M_1' and M_2'.

$$M_1' = a_i + b_i + c_i + d_i, \quad M_2' = b_i + c_i + d_i + e_i$$

Step II.

Job	M_1'	M_2'
1	21 (8 + 4 + 6 + 3)	22 (4 + 6 + 3 + 9)
2	22 (7 + 6 + 4 + 5)	25 (6 + 4 + 5 + 10)
3	16 (6 + 5 + 3 + 2)	18 (5 + 3 + 2 + 8)
4	16 (9 + 2 + 1 + 4)	13 (2 + 1 + 4 + 6)

Step III. The optimal sequence can be determined as minimum of processing time of 13 occurs on M_2' for job 4 it will be processed last. Next minimum time is for job 3 on machine M_1' so it will be processed earliest. Next shortest time is for machine 1 on M_1', so it will be sequenced next to job 3 and so on.

3	1	2	4

Step IV. Total time can be calculated with the hlp of the matrix shown below.

Job	M_1		M_2		M_3		M_4		M_5	
	In	**Out**	**In**	**Out**	**In**	**Out**	**In**	**Out**	**In**	**Out**
1	0	8	8	12	12	18	18	21	21	30
2	8	15	15	21	21	25	25	30	30	40
3	15	21	21	26	26	29	29	32	40	48
4	21	30	30	32	30	31	32	36	48	54

Hence total minimum elapsed time is 51.

Idle time for machines M_1 = 24 hours

$M_2 = 3 + 4 + 22 = 29$

NOTES

$M_3 = 3 + 1 + 1 + 23 = 28$

$M_4 = 4 + 18 = 22$

Two Jobs '*m*' Machines Case

1. Two axis to represent job 1 and 2 are drawn at right angles to each other. Same scale is used for X and Y-axes. X-axis represents the processing time and sequence of job 1 and Y-axis represents the processing time and sequence of job 2. The processing time on machines are laid out in the technological order of the problem.
2. The area which represents processing times of jobs 1 and 2 and is common to both the jobs is shaded. As processing of both the jobs on it machine is not feasible, the shaded area represents the unfeasible region in the graph.
3. The processing of both the jobs 1 and 2 are represented by a continued path which consists of horizontal, vertical and 45 degree diagonal region. The path starts at the lower left corner and stops at upper right corner and the shaded area is avoided. The path is not allowed to pass through shaded area which as brought out in step II represents both the jobs being processed simultaneously on the same machine.

   Any vertical movement represents that job 2 is in progress and job 1 is waiting to be processed. Horizontal movement along the path indicates that job 1 is in progress and job 2 is idle waiting to be processed. The diagonal movement of the path indicates that both the jobs are being processed on different machines simultaneously.
4. A feasible path maximizes the diagonal movement minimizes the total processing time.
5. Minimum elapsed time for any job = processing time of the job + idle time of the same job.

Example 13.7. *The operation time of two jobs 1 and 2 on 5 machines M_1, M_2, M_3, M_4 and M_5 is given in the following table. Find out the optimum sequence in which the jobs should be processed so that the total time used is minimum. The technological order of use of machine for job 1 is M_1, M_2, M_3, M_4 and M_5 for job 2 is M, M_1, M_4, M_5 and M_2.*

Time Hours

Job					
Job 1	M_1	M_2	M_3	M_4	M_5
	1	2	3	5	1
Job 2	M_3	M_1	M_4	M_5	M_2
	3	4	2	1	5

Job 1 preceeds job 2 on machine M_1, job 1 preceeds job 2 on machine M_2, job 2 preceeds job 1 on machine M_3, job 1 preceeds j and job 2 preceeds job 1 on M_5.

The minimum processing time for jobs 1 and 2, total processing time for job 1 + idle time for Job 1 = 12 + 3 = 15 hours

Total processing time for job 2 + idle tim for job 2 = 15 + 0 = 15 hours.

NOTES

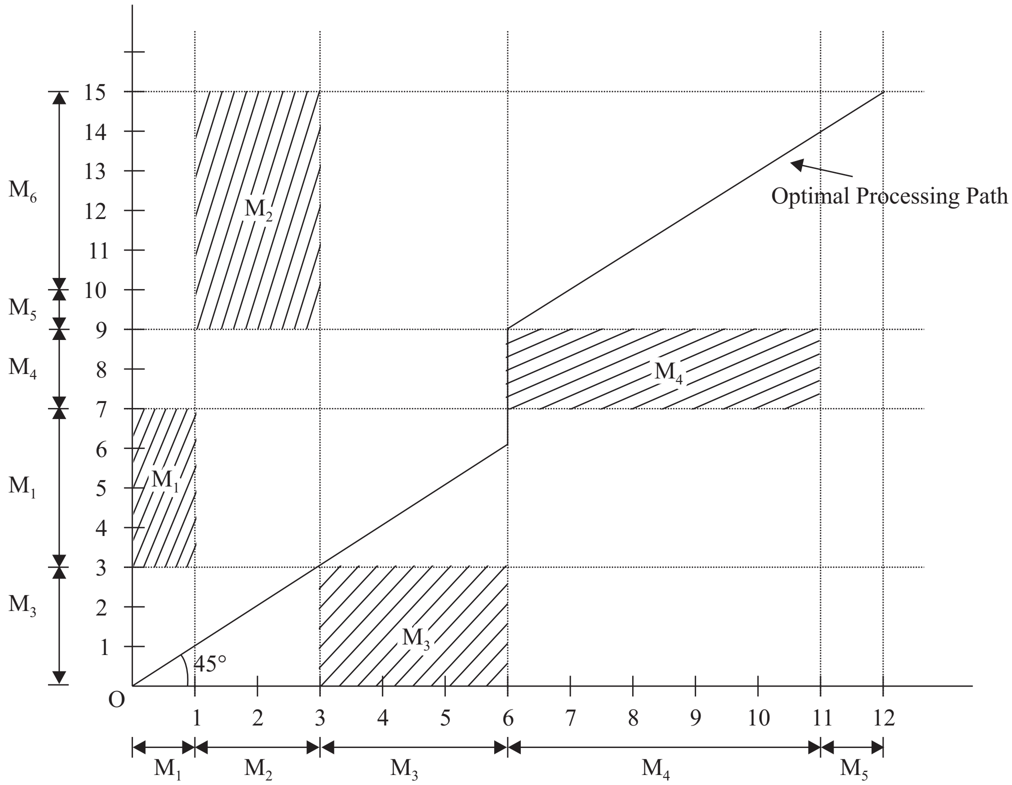

Fig. 9.3

Example 13.8. *Use geographical method to minimize time added to process the following jobs on the machines shown, i.e., for each machine find the job which should be done first. Also calculate the total time elapsed to complete both the jobs.*

Job 1	Sequence	A	B	C	D	E
	Time	3	4	2	6	2
Job 2	Sequence	B	C	A	D	E
	Time	5	4	3	2	6

Solution. The information provided in the problem can be used to draw the following diagram. The shaded area is of the overlap and need to be avoided.

The path that minimizes the idle time for Job 1 is an optimal path. Also the ideal (optimal) path should minimize the idle time for Job 2. For working out the elapsed time, we have to add the idle time for either of the two jobs to that time which is taken for processing of that job. It can be seen that idle time for the chosen path for Job 1 is 5 hours and for Job 2 it is 2 hours, the elapsed time can be calculated as

Processing time for Job 1 + idle time for Job 1 = 17 + (2 + 3) = 22 hours

Processing time for Job 2 + idle timor Job 2 = 20 + 2 = 22 hours.

NOTES

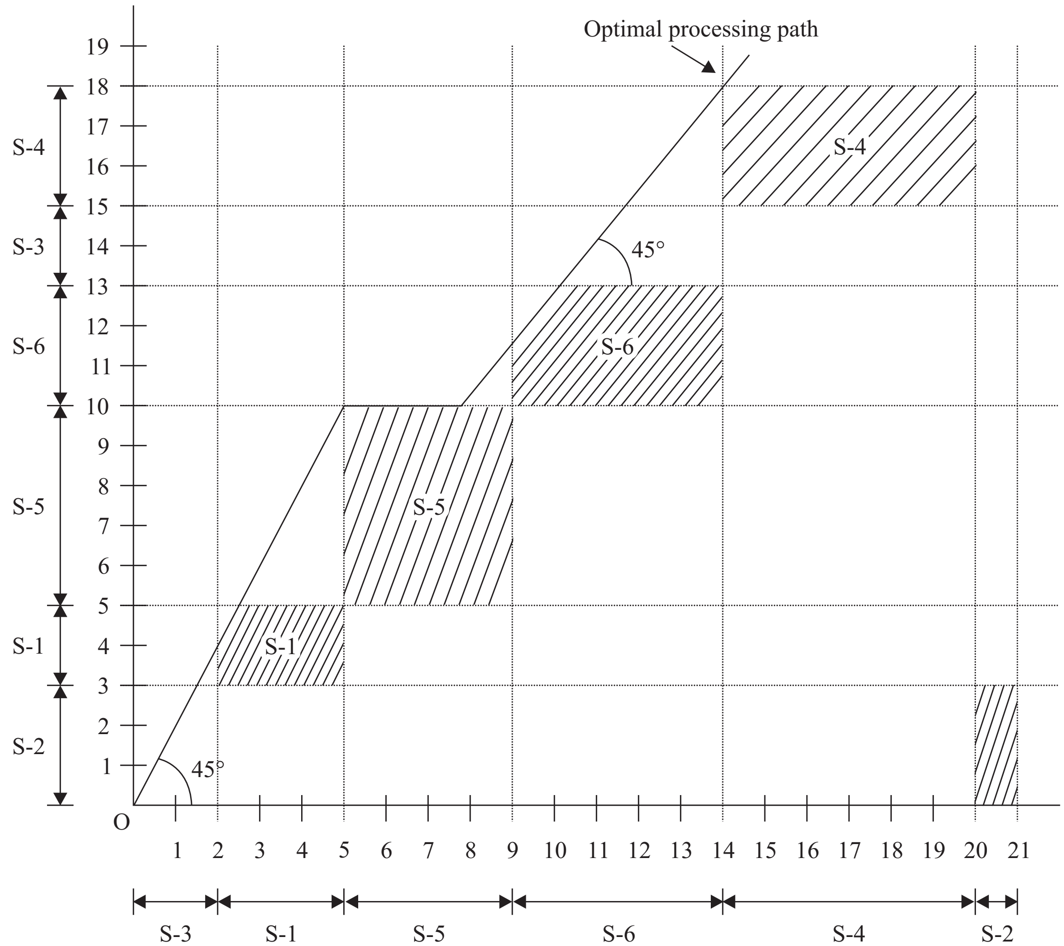

Fig. 9.4

9.3 SUMMARY

- A sequence is the order in which different jobs are to be performed. When there is a choice that a number of tasks can be performed in different orders, then the problem of sequencing arises.
- The basic concept behind sequencing is to use the available facilities in such a manner that the cost (and time) is minimized. The sequencing theory has been developed to solve difficult problems of using limited number of facilities in an optimal manner to get the best production and minimum costs.
- **Job:** These have to be sequenced, hence there should be a particular number of jobs (groups of tasks to be performed) say n to be processed.
- **Machine:** Jobs have to be performed or processed on machines. It is a facility which has some processing capability.
- **Loading:** Assigning of jobs to facilities and committing of facilities to jobs without specifying the time and sequence.
- **Scheduling:** When the time and sequence of performing the job is specified, it is called *scheduling*.

- **Total Elapsed Time:** It is the time that lapses between the starting of first job and the completion of the last one.
- **Idle Time:** The time for which the facilities or machine are not utilized during the total elapsed time.
- **Static arrival Pattern:** If all the jobs to be done are received at the facilities simultaneously.
- **Dynamic arrival Pattern:** Here the jobs keep arriving continuously.

9.4 REVIEW AND DISCUSSION QUESTIONS

1. What is no passing rule in a sequencing algorithm ?
2. Explain the four elements that characterize a sequencing problem.
3. Explain the principal assumptions made while dealing with sequencing problems.
4. Describe the method of processing '*n*' jobs through two machines.
5. Give Johnson's procedure for determining an optimal sequence for processing *n* items on two machines. Give justification of the rules used in the procedure.
6. Explain the method of processing '*m*' jobs on three machines A, B, C in the order ABC.
7. Explain the graphical method to solve the two jobs *m*-machines sequencing problem with given technological ordering for each job. What are the limitations of the method ?
8. A Company has 8 large machines, which receive preventive maintenance. The maintenance team is divided into two crews A and B. Crew A takes the machine 'Power' and replaces parts according to a given maintenance schedule. The second crew resets the machine and puts it back into operation. At all times 'no passing' rule is considered to be in effect. The servicing times fr each machine are given below.

Machine	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*
Crew A	5	4	22	16	15	11	9	4
Crew B	6	10	12	8	20	7	2	21

Determine the optimal sequence of scheduling the factory maintenance crew to minimize their idle time and represent it on a chart.

9. Use graphical method to find the minimum elapsed total time sequence of 2 jobs and 5 machines, when we are given the following information :

Job 1	Sequence	A	B	C	D	E
	Time (hours)	2	3	4	6	2
Job 2	Sequence	C	A	D	E	B
	Time (hours)	4	5	3	2	6

NOTES

10. Two jobs are to be processed on four machines *a*, *b*, *c* and *d*. The technological order for these obs on machines is as follows :

Job 1	*a*	*b*	*c*	*d*
Job 2	*d*	*b*	*a*	*c*

Processing times ar given in the following table :

	Machines			
Job	*a*	*b*	*c*	*d*
1	4	6	7	3
2	4	7	5	8

Find the optimal sequence of jobs on each of machines.

11. A machine shop has four machines A, B, C and D. Two jobs must be processed through each of these machines . The time (in hours) taken on each of these machines and the necessary sequence of jobs trough the shop are given below.

Job 1	Sequence	A	B	C	D
	Time (hours)	2	4	5	1
Job 2	Sequence	D	B	C	A
	Time (hours)	6	4	2	3

Use graphic method to obtaine total minimum elapsed time.

UNIT 10: WAITING LINE (QUEUING) THEORY

Structure

10.1 INTRODUCTION

Queuing theory has been used for many real life applications to a great advantage. It is so because many problems of business and industry can be assumed/simulated to be arrival-departure or queuing problems. In any practical life situations, it is not possible to accurately determine the arrival and departure of customers when the number and types of facilities as also the requirements of the customers are not known. Queuing theory techniques, in particular, can help us to determine suitable number and type of service facilities to be provided to different types of customers. Queuing theory techniques can be applied to problems such as:

(*a*) Planning, scheduling and sequencing of parts and components to assembly lines in a mass production system.

(*b*) Scheduling of workstations and machines performing different operations in mass production.

(*c*) Scheduling and dispatch of war material of special nature based on operational needs.

(*d*) Scheduling of service facilities in a repair and maintenance workshop.

(*e*) Scheduling of overhaul of used engines and other assemblies of aircrafts, missile systems, transport fleet, etc.

(*f*) Scheduling of limited transport fleet to a large number of users.

(*g*) Scheduling of landing and take-off from airports with heavy duty of air traffic and limited facilities.

(*h*) Decision of replacement of plant, machinery, special maintenance tools and other equipment based on different criteria.

Special **benefit** which this technique enjoys in solving problems such as above are:

(*i*) Queuing theory attempts to solve problems based on a scientific understanding of the problems and solving them in optimal manner so that facilities are fully utilised and waiting time is reduced to minimum possible.

(*ii*) Waiting time (or queuing) theory models can recommend arrival of customers to be serviced, setting up of workstations, requirement of manpower, etc., based on probability theory.

NOTES

Limitation of Queuing theory

Though queuing theory provides us a scientific method of understanding the queues and solving such problems, the theory has certain limitations which must be understood while using the technique, some of these are :

(*a*) Mathematical distributions, which we assume while solving queuing theory problems, are only a close approximation of the behaviour of customers, time between their arrival and service time required by each customer.

(*b*) Most of the real life queuing problems are complex situation and are very difficult to use the queuing theory technique, even then uncertainty will remain.

(*c*) Many situations in industry and service are multi-channel queuing problems. When a customer has been attended to and the service provided, it may still have to get some other service from another service point and may have to fall in queue once again. Here the departure of one channel queue becomes the arrival of the other channel queue. In such situations, the problem becomes still more difficult to analyse.

(*d*) Queuing model may not be the ideal method to solve certain very difficult and complex problems and one may have to resort to other techniques like Monte-Carlo simulation method.

10.2 IMPORTANT TERMS USED IN QUEUING THEORY

Following are some important terms used in queuing theory:

1. **Arrival Pattern:** It is the pattern of the arrival of a customer to be serviced. The pattern may be regular or at random. Regular interval arrival patterns are rare, in most of the cases, arrival of the customers cannot be predicted. Regular pattern of arrival of customers follows Poisson's distribution.

2. **Poisson's Distribution:** It is discrete probability distribution which is used to determine the number of customers in a particular time. It involves allotting probability of occurrence of the arrival of a customer. Greek letter l (lamda) is used to denote mean arrival rate. A special feature of the Poisson's distribution is that its mean is equal to the variance. It can be represented with the notation as explained below.

$P(n)$ = Probability of n arrivals (customers)

l = Mean arrival rate

e = Costnt = 2.71828

$$P(n) = \frac{e^{-\lambda} (\lambda)^n}{\lfloor n}, \text{ where } n = 0, 1\ 2, ...$$

Notation or $\lfloor\ $ is called the factoral and it means that

$\lfloor n$ or $n! = n(n-1)(n-2)(n-3)$ 2, 1

Poisson's distribution tables for different values of n is available and can be used for solving problems where Poisson's distribution is used. However, It has certain limitations because of which its use is restricted. It assumes that arrivals are random and independent of all other variables or parameters. Such can never be the case.

3. **Exponential Distribution:** This is based on the probability of completion of a service and is the most commonly used distribution in queuing theory. In queuing theory, our effort is to minimize the total cost of queue and it includes cost of waiting and cost of providing service. A queue model is prepared by taking different variables into consideration. In this distribution system, no maximization or minimization is attempted. Queue models with different alternatives are considered and the most suitable for a particular is attempted. Queue models with different alternatives are considered and the most suitable for a particular situation is selected.
4. **Service Pattern:** We have seen that arrival pattern is random and poissons distribution can be used for use in queue model. Service pattern are assumed to be exponential for purpose of avoiding complex mathematical problem.
5. **Channels:** A service system has a number of facilities positioned in a suitable manner. These could be

   (a) *Single Channel Single Phase System.* This is very simple system where all the customers wait in a single line in front of a single service facility and depart after service is provided. In a shop if there is only one person to attend to a customer, is an example of the system.

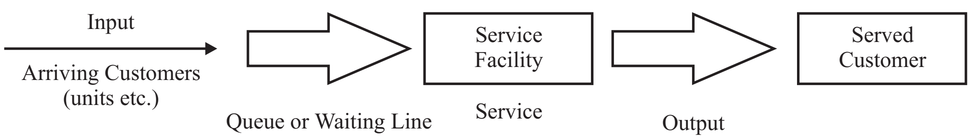

Fig. 12.1

   (b) Service in series: Here the input gets serviced at one service station and then moves to second and or third and so on before going out. This is the case when a raw material input has to undergo a number of operations like cutting, turning drilling etc.

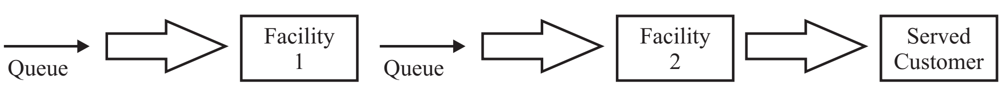

Fig. 12.2

   (c) Multi-parallel facility with a single queue: Here the service can be provided at a number of points to one queue. This happens when in a grocery store, there are 3 persons servicing the same queue or a service station having more than one facility of washing cars. This is shown in figure 9.3

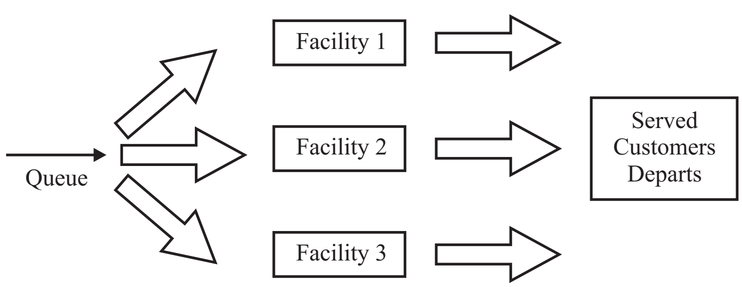

Fig. 12.3

   (d) Multiple parallel facilities with multiple queue: Here there are a number of queues and separate facility to service each queue. Booking of tickets at railway stations, bus stands, etc., is a good example of this. his is shown in figure 9.4.

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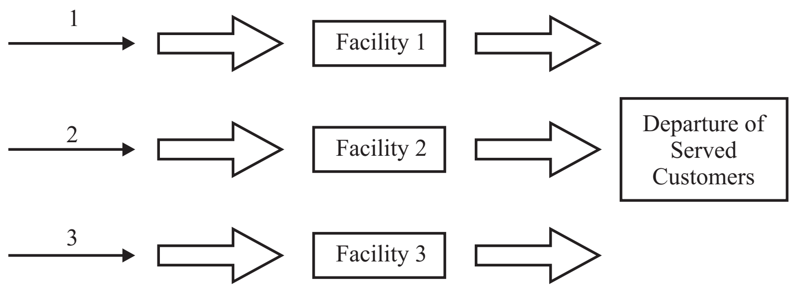


Fig. 12.4

6. **Service Time:** Service time, i.e., the time taken by the customer when the facility is dedicated to it for serving depends upon the requirement of the customer and what needs to be done as assessed by the facility provider. The arrival pattern is random so also is the service time required by different customers. For the sake of simplicity the time required by all the customers is considered constant under the distribution. If the assumption of exponential distribution is not valid, Erlang Distribution is applied to the queuing model.
7. **Erlang Distribution:** It has been assumed in the queuing process we have seen that service is either constant or it follows negative exponential distribution in which case the standard deviation s (sigma) is equal to its mean. This assumption makes the use of the exponential distribution simple. However, in cases where s and mean are not equal, Erlang distribution developed by AK Erlang is used. In this method, the service time is divided into number of phases assuming that total service can be provided by different phases of service. It is assumed that service time of each phase follows the exponential distribution, i.e., σ = mean.
8. **Traffic Intensity or Utilisation Rate:** This is the rate of at which the service facility is utilised by the components.

   If l = mean arrival rate and

   (Mue) μ = Mean service rate, then utilisation rate (p) = l/m it can be easily seen from the equation that $p > 1$ when arrival rate is more than the service rate and new arrivals will keep increasing the queue. $p < 1$ means that service rate is more than the arrival rate and the waiting time will keep reducing as m keeps increasing. This is true from the commonsense.
9. **Idle Rate.** This is the rate at which the service facility remains unutilised and is lying idle.

   Idle rate = 1 – utilisation rate = $1 - p = \left(1 - \frac{\lambda}{\mu}\right) \times$ total service facility = $.\left(1 - \frac{\lambda}{\mu}\right) \times \frac{\lambda}{\mu}$
10. **Expected number of customers in the system.** This is the number of customers in queue plus the number of customers beig serviced and is denoted by $E_n = \frac{\lambda}{(\mu - \lambda)}$.
11. **Expected number of customers in queue (Average queue length).** This is the number of expected customers minus the number being serviced ad i denoted by E_q.

$$E_q = E_n - p = \frac{\lambda}{(\mu - \lambda)} - \frac{\lambda}{\mu} = \frac{\lambda^2}{\mu(\mu - \lambda)}$$

12. **Expected time spent by customer in system.** It is the time that a customer spends waiting in queue plus the time it takes for servicing the customer and is denoted by E_t

$$E_t = \frac{E_n}{\lambda} = \frac{\frac{\lambda}{(\mu - \lambda)}}{\lambda} = \frac{1}{(\mu - \lambda)}.$$

13. **Expected waiting time in queue.** It is known that Et = expected waiting time in queue + expected service time, therefore expected waiting time in queue (E_w) = $E_t - \frac{1}{\mu}$.

14. **Average length of non-empty queue**. $E_l = .\frac{\mu}{(\mu - \lambda)} = \frac{1}{(\mu - \lambda)} - \frac{1}{\mu} = \frac{\lambda}{\lambda(\mu - \lambda)}$

15. **Probability that customer wait is zero.** It means that the customer is attended to for servicing at the point of arrival and the customer does not wait at all. This depends upon the utilization rate of the service or idle rate of the system, p_0 = 0 persons waiting in the queue = $1 - \frac{\lambda}{\mu}$ and the probability of 1, 2, 3 .. , *n* persons waiting in the queue will be given by

$$p_1 = p_0\left(\frac{\lambda}{\mu}\right)^1, p_2 = p_1\left(\frac{\lambda}{\mu}\right)^2, p_n = p_0\left(\frac{\lambda}{\mu}\right)^n.$$

16. **Queuing Discipline.** All the customers get into a queue on arrival and are then serviced. The order in which the customer is selected for servicing is known as queuing discipline. A number of systems are used to select the customer to be served. Some of these are :
    (*a*) *First in First Served (FIFS):* This is the most commonly used method and the customers are served in the order of their arrival.
    (*b*) *Last in First Served (LIFS):* This is rarely used as it will create controversies and ego problems amongst the customers. Any one who comes first expects to be served first. It is used in store management, where it is convenient to issue the store last received and is called Last In First Out (LIFO).
    (*c*) *Service in Priority (SIP):* The priority in servicing is allotted based on the special requirement of a customer like a doctor may attend to a serious patient out of turn, so may be the case with a vital machine which has broken down. In such cases the customer being serviced may be put on hold and the priority customer attended to or the priority may be on hold and the priority customer waits till the servicing of the customer already being serviced is over.

17. **Customer Behaviour:** Different types of customers behave in different manner while they are waiting in queue, some of the patterns of behaviour are :
    (*a*) *Collusion*: Some customers who do not want to wait they make one customer as their representative and he represents a group of customers. Now only the representative waits in queue and not all members of the group.
    (*b*) *Balking*: When a customer does not wait to join the queue at the correct place which he warrants because of his arrival. They want to jump the queue and move ahead of others to reduce their waiting time in the queue. This behaviour is called balking.

NOTES

(*c*) *Jockeying*: This is the process of a customer leaving the queue which he had joined and goes and joins another queue to get advantage of being served earlier because the new queue has lesser customers ahead of him.

(*d*) *Reneging*: Some customers either do not have time to wait in queue for a long time or they do not have the patience to wait, they leave the queue without being served.

18. **Queuing Cost Behaviour.** The total cost a service provider system incurs is the sum of cost of providing the services and the cost of waiting of the customers. Suppose the garage owner wants to install another car washing facility so that the waiting time of the customer is reduced. He has to manage a suitable compromise in his best interest. If the cost of adding another facility is more than offset by reducing the customer waiting time and hence getting more customers, it is definitely worth it. The relationship between these two costs is shown below.

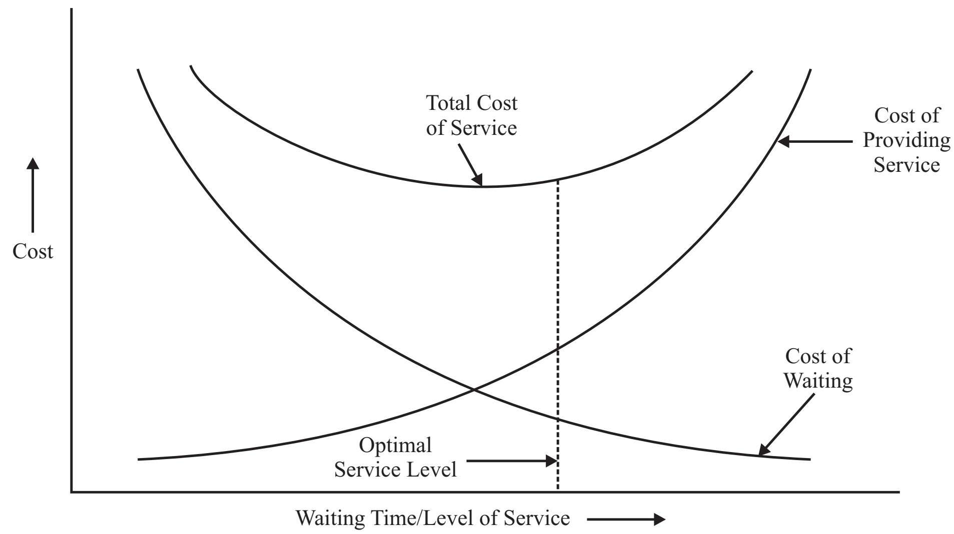


Fig. 12.5

10.4 TYPES OF QUEUING MODELS

Different types of models are in use. The three possible types of categories are :

(*a*) ***Deterministic model*:** Where the arrival and service rates are known. This is rarely used as it is not a practical model.

(*b*) ***Probabilistic model***: Here both the parameters, *i.e.*, the arrival rate as also the service rate are unknown and are assumed random in nature. Probability distribution, *i.e.*, Poissons, Exponential or Erlang distributions are used.

(*c*) ***Mixed model*:** Where one of the parameters out of the two is known and the other is unknown.

10.5 SINGLE CHANNEL QUEUING MODEL

(Arrival — Poisson and Service time Exponential)

This is the simplest queuing model and is commonly used. It makes the following assumptions:

(*a*) Arriving customers are served on First Come First Serve (FCFS) basis.

(*b*) There is no Balking or Reneging. All the customers wait the queue to be served, no one jumps the queue and no one leaves it.

(*c*) Arrival rate is constant and does not change with time.

(*d*) New customers arrival is independent of the earlier arrivals.

(*e*) Arrivals are not of infinite population and follow Poisson's distribution.

(*f*) Rate of serving is known.

(*g*) All customers have different service time requirements and are independent of each other.

(*h*) Service time can be described by negative exponential probability distribution.

(*i*) Average service rate is higher than the average arrival rate and over a period of time the queue keeps reducing.

Example 10.1. *Assume a single channel service system of a library in a school. From past experiences it is known that on an average every hour 8 students come for issue of the books at an average rate of 10 per hour. Determine the following:*

(a) Probability of the assistant librarian being idle.

(b) Probability that there are at least 3 students in system.

(c) Expected time that a student is in queue.

Solution.

(a) Probability that server is idle = $\left(\frac{\lambda}{\mu}\right)\left(1-\frac{\lambda}{\mu}\right)$ in this example $\lambda = 8$, $\mu = 10$

$$p_0 = \frac{8}{10}\left(1-\frac{8}{10}\right) = 16\% = 016.$$

(b) Probability that at least 3 students are in the system

$$E_n = \left(\frac{\lambda}{\mu}\right)^{3+1} = \left(\frac{8}{10}\right)^4 = 0.4$$

(c) Expected time that a students is in queue

$$\frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{64}{(10\times 2)} = 3.2 \text{ hours.}$$

Example 10.2. *Self-help canteen employs one cashier at its counter, 8 customers arrive every 10 minutes on an average. The cashier can serve at the rate of one customer per minute. Assume Poisson's distribution for arrival and exponential distribution for service patterns. Determine*

(a) Average number of customers in the system;

(b) Average queue length;

(c) Average time a customer spends in the system.

Solution. Arrival rate $\lambda = \frac{8}{10}$ customers/minute

Service rate $\mu = 1$ customer/minute

(a) Average number of customers in the system

$$E_n = \frac{\lambda}{\mu - \lambda} = \frac{0.8}{1 - 0.8} = 4$$

(b) Average queue length

NOTES

$$E_q = \frac{\lambda^2}{\mu\ (\mu - \lambda)} = \frac{(0.8)^2}{1 \times 0.2} = 3 \times 2.$$

(c) Average time a customer spends in the queue

$$E_w = \frac{\lambda}{\mu\,(\mu - \lambda)} = \frac{0.8}{1 \times 0.2} = 4 \text{ minutes.}$$

Example 10.3. *Arrival rate of telephone calls at telephone booth are according to Poisson distribution, with an average time of 12 minutes between two consecutive calls arrival. The length of telephone calls is assumed to be exponentially distributed with mean 4 minutes.*

(a) Determine the probability that person arriving at the booth will have to wait.

(b) Find the average queue length that is formed from time to time.

(c) The telephone company will install second booth when convinced that an arrival would expect to have to wait at least 5 minutes for the phone. Find the increase in flows of arrivals which will justify a second booth.

(d) What is the probability that an arrival will have to wait for more than 15 minutes before the phone is free ?

(e) Find the fraction of a day that the phone will be in use.

Solution. Arrival rate λ = 1/12 minutes

Service rate μ = 1/4 minues.

(a) Probability that a person will have to wait = $\frac{\lambda}{\mu} = \frac{\frac{1}{12}}{\frac{1}{4}} = \frac{1}{12} \times 4 = \frac{1}{3} = 0.33$

(b) Average queue length = $E_q = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{\frac{1}{144}}{\frac{1}{4}\left(\frac{1}{4} - \frac{1}{12}\right)} = \frac{1}{144} \times 4 \times \frac{12}{2} = 1$ person.

(c) Average waiting time in the queue $E_w = \frac{\lambda_1}{\mu(\mu - \lambda_1)} = \frac{\lambda_1}{\frac{1}{4}(\mu - \lambda_1)}$

$$5 = \frac{\lambda_1}{\frac{1}{4}\left(\frac{1}{4} - \lambda_1\right)}, \quad \frac{5}{16} = \left(\frac{5}{4} + 1\right)\lambda_1$$

$$\lambda_1 = \frac{5}{16} \times \frac{4}{9} = \frac{5}{36} \text{ arrivals/minute}$$

Increase in flow of arrivals = $\frac{5}{36} - \frac{1}{12} = \frac{1}{18}$ minutes

(d) Probability of waiting time > 15 minutes.

$$= \frac{\lambda}{\mu} e^{(\lambda - \mu)15} = \frac{\frac{1}{12}}{\frac{1}{4}} e^{\left(\frac{1}{12} - \frac{1}{4}\right)15} = \frac{1}{3} e^{-\frac{30}{12}} = \frac{1}{3} e^{-2.5}$$

(e) Fraction of a day that phone will be in use = $\frac{\lambda}{\mu} = 0.33$.

Example 12.4 *An electricity bill receiving window in a small town has only one cashier who handles and issues receipts to the customers. He takes on an average 5 minutes per customer. It has been estimated that the persons coming for bill payment have no set pattern but on an average 8 persons come per hour. The management receives a lot of complaints regarding customers waiting for long in queue and so decided to find out.*

(a) What is the average length of queue ?

(b) What time on an average, the cashier is idle ?

(c) What is the average time for which a person has to wait to pay his bill ?

(d) What is the probability that a person would have to wait for at least 10 minutes ?

Solution. Making use of the usual notations

$$\lambda = 8 \text{ person/hour}$$

$$\mu = 10 \text{ persons/hour}$$

(a) Average queue length = $\frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{64}{10(10-8)}$ = 32 persons

(b) Probability that cashier is idle = $p_0 = 1 - \frac{\lambda}{\mu} = 1 - \frac{8}{10} = 0.2$, *i.e.*, the cashier would be idle for, 20 % of his time.

(c) Average length of time that a person is expected to wait in queue.

$$E_w = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{8}{10(10-8)} = 24 \text{ minutes}$$

(d) Probability that a customer will have to wait for at least 10 minutes.

$$p(8) = \frac{\lambda}{\mu} \times e^{-(8-10)\times\frac{1}{6}} = \frac{8}{10} e^{-33}, t = \frac{1}{6} \text{ hours.}$$

Example 10.5. *ABC Diesel engineering works gets on an average 40 engines for overhaul per week, the need of getting a diesel engine overhauled is almost constant and the arrival of the repairable engines follows Poissons's distribution.*

However, the repair or overhaul time is exponentially distributed. An engine not available for use costs Rs. 500 per day. There are six working days and the company works for 52 weeks per year. At the moment the company has established the following overhaul facilities.

	Facilities	
	1	2
Installation Charges	1200000	1600000
Operating Expenses / year	200000	350000
Economic life (years)	8	10
Service Rate/Week	50	80

The facilities scrap value may be assumed to be nil. Determine which facility should be preferred by the company, assuming time value of money is zero ?

NOTES

Solution. Let us work out the total cost of using both the facilitates.

Facility 1: λ = 40/week, μ = 50/week

Total annual cost = Annual capital cost + Annual operating cost + Annual cost of lost time of overhaul able engines.

Expected annual lost time = (Expected time spent by repairable engines in sste) × (Expected number of arrivals in a year).

$$E_t = \frac{1}{\mu-\lambda}(\lambda \times \text{ number of weeks}) = \frac{1}{(50-40)} \times 40 \times 52 = 208 \text{ weeks.}$$

Cost of the lost time = Rs. 208 × 6 × 500 = 624000

Total annual cost

$$= \frac{1200000}{8} + 200000 + 624000 = 150000 + 200000 + 64000$$

$$= \text{Rs. } 974000$$

Facility 2: Annual capital cost

$$= \frac{1600000}{10} + 350000 + \text{cost of lost engine availability time}$$

$$\text{Cost o lost availability time} = E_t \times (\lambda \times \text{number of weeks}) = \frac{1}{(\mu-\lambda)} \times (\lambda \times \text{number of weeks})$$

Here $\lambda = 40$

$\mu = 80$

$$\text{Hence, cost of lost availability time} = \frac{1}{80-40} \times (40 \times 52) = \frac{2080}{40} = 52 \text{ weeks/years.}$$

Cot of lost time = 52 × 6 × 52 = Rs. 162245

$$\text{Total cost} = \frac{1600000}{10} + 350000 + 162245 = \text{Rs. } 672245$$

Hence, facility No. 2 should be preferred to facility number one.

10.3 MULTI-CHANNEL QUEUING MODEL (ARRIVAL POISSON AND SERVICE TIME EXPONENTIAL)

This is a common facilities system used in hospitals or banks where there are more than one service facilities and the customers arriving for service are attended to by these facilities on first come first serve basis. It amounts to parallel service points in front of which there is a queue. This shortens the length of the queue if there was only one service station. The customer has the advantage of shifting from a longer queue where he has to spend more time to shorter queue and can be serviced in lesser time. Following assumptions are made in this model:

(a) The input population is infinite, i.e., the customers arrive out of a large number and follow Poisson's distribution.

(b) Arriving customers form one queue.

(c) Customer are served on First come First served (FCFS) basis.

(d) Service time follows an exponential distribution.

(e) There are a number of service station (K) and each one provides exactly the same service.

(f) The service rate of all the service stations put together is more than arrival rate.

In this analysis we will use the following notations.

λ = Average rate of arrival

μ = Average rate of service of each of the service stations

K = Number of service stations

Kμ = Mean combined service rate of all the service stations.

Hence ρ(*rho*) the utilisation factor for th system = $\dfrac{\lambda}{K\mu}$.

(a) Probability that system will be idle $p_0 = \left[\sum_{n=0}^{K-1} \frac{\left(\frac{\lambda}{\mu}\right)^n}{\lfloor n} + \frac{\left(\frac{\lambda}{\mu}\right)^k}{\lfloor k(1-\rho)}\right]^{-1}$

(b) Probability of *n* customers in the system.

$$\rho_n = \frac{\left(\frac{\lambda}{\mu}\right)^n}{\lfloor n} \times \rho_0 n \le k$$

$$\rho_n = \frac{\left(\frac{\lambda}{n}\right)^n}{\lfloor k} K^{n-k} \times \rho_0 \quad n > k$$

(c) Expected number of customers in queues or queue length

$$E_q = \frac{\left(\frac{\lambda}{\mu}\right)^k \rho}{\lfloor k(1-l)^2} \times \rho_0.$$

(*d*) Expected number of customers in the system = $E_n = E_q + \dfrac{\lambda}{\mu}$

(e) Average time a customer spends in queue

$$E_w = \frac{E_q}{\lambda}$$

(f) Average time a customer spends in waiting line

$$= E_w + .\frac{1}{\mu}$$

Example 10.6. *A workshop engaged in the repair of cars has two separate repair lines assembled and there are two tools stores one for each repair line. Both the stores keep in identical type of tools. Arrival of vehicle mechanics has a mean of 16 per hour and follows a Poisson distribution. Service time has a mean of 3 minutes per machine and follows an exponential distribution. Is it desirable to combine both the tool stores in the interest of reducing waiting time of the machie and improving the efficiency?*

Solution. λ = 16/hour

$$\mu = 1\frac{1}{3} \times 60 = 20 \text{ hours}$$

Expected waiting time in queue, $E_w = \dfrac{\lambda}{\mu(\mu-\lambda)} = \dfrac{16}{20(20-16)}$ = 0 . 2 hour = 12 minutes. If we combine the two tools stores.

$\lambda =$ Mean arrival rate = 16 + 16 = 32 / hour K = 2, n = 1.

$\mu =$ Mean service rate 20/our

NOTES

Expected waiting time in queue, $E_w = \dfrac{E_q}{\lambda} = \dfrac{\lambda\mu\left(\frac{\lambda}{\mu}\right)^k}{\underline{|k-1}(K\mu-\lambda)^2} \times \rho_0$.

$$\text{where} \quad \rho_0 = \left[\sum_{n=0}^{k-1} \frac{\left(\frac{\lambda}{\mu}\right)^k}{\underline{|k}\left\{1-\dfrac{\lambda}{k\mu}\right\}}\right]^{-1}$$

$$= \left[\sum_{n=0}^{1} \frac{\left(\frac{32}{20}\right)^n}{\underline{|n}} + \frac{\left(\frac{32}{20}\right)^2}{\underline{|2}-\left\{1-\frac{32}{2\times 24}\right\}}\right]^{-1}$$

$$= 0.182$$

$$E_w = \frac{E_q}{\lambda} \times p_0$$

$$\frac{E_q}{\lambda} = \frac{32\left(\frac{32}{20}\right)}{\underline{|2-1}(40-32)^2} = \frac{32}{25}$$

$$\text{Hence} \quad E_w = \frac{32}{25} \times 0.182 = 14 \text{ minutes.}$$

Since the waiting time in queue has increased, it is not desirable to combine both the tools stores. Present system is more efficient.

Example 10.7. *A bank has three different single window service counters. Any customer can get any service from any of the three counters. Average time of arrival of customer is 12 per hour and it follows Poisson's distribution. Also, on average the bank officer at the counter takes 4 minutes for servicing the customer. The bank is considering the option of installing ATM, which is expected to be more efficient and service the customer twice as the bank officers do at present. If the only consideration of the bank is to reduce the waiting time of the customer, which system is better ?*

Solution. The existing system is multi-channel system, using the normal notations here

$$\lambda = 12 / \text{hour} = \frac{60}{4} = 15 / \text{hour}$$

Average time a customer spends in the queue waiting to be served.

E_q = Average number of customer in he queue waiting t be served.

$$E_q = \frac{\lambda\mu\left(\frac{\mu}{\mu}\right)^k}{\underline{|k-1}(k\mu-\lambda)^2} \times \rho_0$$

$$\text{Or} \qquad E_w = \frac{E_q}{\lambda} = \frac{\mu\left(\frac{\lambda}{\mu}\right)^k}{\underline{|k-1}(k\mu-\lambda)^2} \times \rho_0$$

$$\text{where} \quad r_0 = \left[\sum_{n=0}^{k=1} \frac{\frac{\lambda}{\mu}}{\underline{|k}} + \frac{\left(\dfrac{\lambda}{\mu}\right)^k}{\underline{|k}\left\{1-\dfrac{\lambda}{k\mu}\right\}}\right]^{-1}$$

NOTES

Here $k = 3$

$$\rho_0 = \left[1 + \frac{12}{15 \times 6} + \frac{\left(\frac{16}{25}\right)^3}{6\left\{1 - \frac{12}{45}\right\}}\right]^{-1}$$

$$\rho_0 = [1 + 0.133 + 0.06]^{-1} = [1.193]^{-1} = 0.83$$

$$E_w = \frac{15\left(\frac{12}{15}\right)^3}{\lfloor 2 (18)^2} \times \rho_0 = 15 \times \frac{64}{(125 \times 2 \times 324)} \times \rho_0$$

$$= 15 \times 64 \times \frac{0.83}{(250 \times 324)} = 0.009 \text{ hour}$$

$$= 0.33 \text{ seconds.}$$

Proposed System

$$E_w = \frac{\lambda}{\mu(\mu - \lambda)} \text{ here } \lambda = 12/\text{hour}, \mu = 15/\text{hour}, E_w = \frac{12}{15(15-12)} = \frac{12}{45} \times 60$$

$$= 16 \text{ minutes}$$

Hence, it is better to continue with the present system rather than installing ATM purely on the consideration of customer waiting time.

Example 10.8. *At a polyclinic three facilities of clinical laboratories have been provided for blood testing. Three lab technicians attend to the patients. The technicians are equally qualified and experienced and they take 30 minutes to serve a patient. This average time follows exponential distribution. The patients arrive at an average rate of 4 per hour and this follows Poisson's distribution. The management is interested in finding out the following :*

(a) Expected number of patients waiting in the queue.

(b) Average time that a patient spends at the polyclinic.

(c) Probability that a patient must wait before being served.

(d) Average percentage idle time for each of the lab technicians.

Solution. In this example

$$\lambda = 4/\text{hour}$$

$$\mu = \frac{1}{30} \times 60 = 2/\text{hour}$$

$$K = 3$$

ρ_0 = Probability that there is no patient in the system.

$$= \left[\sum_{n=1}^{k-1} \frac{1}{\lfloor n} \left(\frac{\lambda}{\mu}\right)^n + \frac{1}{\lfloor n} \frac{\left(\frac{\lambda}{\mu}\right)^k}{\left(1 - \frac{\lambda}{k\mu}\right)}\right]^{-1}$$

$$= \left[\frac{1}{\lfloor 0} + \frac{2}{\lfloor 2} + \frac{2^2}{\lfloor 2} + \frac{1}{16}(2)^3 \times \frac{1}{1} - \frac{4}{6}\right]^{-1} = \left[1 + \frac{2^1}{\lfloor 2} + \frac{2^2}{\lfloor 2} + \frac{(2)^3}{2 \times \frac{2}{6}}\right]^{-1}$$

$$= \left[1+1+2\frac{8\times 6}{4}\right]^{-1} = (26)^{-1} = 0.038$$

NOTES

(*a*) Expected number of patients waiting in the queue

$$E_q = \frac{1}{\underline{|k-1}}\left(\frac{\lambda}{\mu}\right)^k \frac{\lambda\mu}{(k\mu-k)^2}\times p_0$$

$$= \left[\frac{1}{2}\times 8\times\frac{8}{4}\right]\times 0.038 = 8\times 0.038 = 0.304 \text{ or one patient}$$

(*b*) Average time a patient spends in the system

$$= \frac{E_q}{\lambda}+\frac{1}{\mu} = \frac{0.304}{4}+\frac{1}{2} = 0.076+0.5 = 0576 \text{ hours} = 35 \text{ minutes}$$

(*c*) Probability that a patient must wait

$$p\,(n\ge k) = \frac{1}{\underline{|k}}\left(\frac{\lambda}{\mu}\right)^k \frac{1}{\left(\frac{1-\lambda}{k\mu}\right)}\times p_0$$

$$= \frac{1}{6}\times 8\times 8\times 0.038$$

$$= 0.40$$

(d) p (idle technician) = $\frac{3}{3}p_0+\frac{2}{3}p_1+\frac{1}{3}p_2$ when $p_n = \frac{1}{\underline{|n}}\left(\frac{\lambda}{\mu}\right)^n p_0$

p_0 = when all the 3 technician are idle (no one is busy)

p_1 = when only one technician is idle (two are busy)

p_2 = when two technicians are idle (only one busy)

$$p\text{ (idle technician)} = \frac{3}{3}\times 0.038+\frac{2}{3}\times\left(\frac{4}{2}\right)\times 0.038+\frac{1}{3}\times\frac{1}{\underline{|2}}(2)^2\times 0.038$$

$$= 0.038 + 0.05 + 0.025$$

$$= 0.113$$

10.4 POISSON ARRIVAL AND ERLANG DISTRIBUTION FOR SERVICE

We have assumed in our earlier problems that the two service pattern distributions follow exponential distribution in a manner that its standard deviation is equal to its mean. But there are many situations where these two will vary, we must use a $\frac{1}{\mu}$ model which is more relevant and applicable to real life situations. In this method the $\frac{1}{\mu}$ servic is considered in a number of phases each with a service time and time taken in e ch phase is exponentially distributed. With same mean time of , with different channels we get different distribution. The method makes the follows assumptions:

(a) The arrival pattern follows Poisson distribution.

(b) One unit completes service in all the phases and only then the other unit is served.

(c) In each phase the service follows exponential distribution.

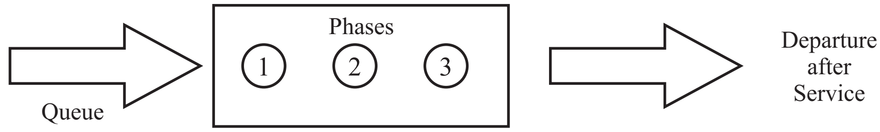


Fig. 12.6

The following formulae are used in t is method:

1. Expected number of customer in the system

$$E_n = k + \frac{1}{2k} \times \frac{\lambda^2}{\mu(\mu - k)} + \frac{\lambda}{\mu} = E_q + \frac{\lambda}{\mu}$$

2. Expected number of customers in the queue (or Average queue length

$$E_q = k + \frac{1}{2k} \times \frac{\lambda^2}{\mu(\mu - \lambda)}$$

3. Average waiting time of a customer in queue

$$E_t = k + \frac{1}{2k} \times \frac{\lambda^2}{\mu(\mu - \lambda)}$$

4. Expected waiting time of a customer in the system

$$E_t = \frac{k+1}{2k} \times \frac{\lambda^2}{\mu(\mu - \lambda)} + \frac{1}{\mu}$$

Example 10.9. *Maintenance of machine can be carried out in 5 operations which have to be performed in a sequence. Time taken for each of these operations has a mean time of 5 minutes and follows exponential distribution. The breakdown of machine follows Poisson distribution and the average rate of breakdown is 3 per hour. Assume that there is only one mechanic available, find out the average idle time for each machine breakdown.*

Solution. K = 3

Arrival $\lambda = \frac{3}{60} = 1/20$ machines/hour

Total service time for one machine = 5 × 3 = 15 minutes

Service rate μ = 1/15 machines/hour

$$\rho = \text{Utilisation rate/traffic intensity} = \frac{\lambda}{k\mu} = \left(\frac{1}{20} \times 3\right) \times 15 = \frac{1}{4} = 0.25$$

Expected idle time for machine $= k + \frac{1}{2} k = \frac{\lambda^2}{\mu(\mu - \lambda)} + \frac{1}{\mu}$

$$= \frac{4}{6} \times \frac{1}{20} \times \frac{1}{20} \times 15\left(\frac{1}{15} - \frac{1}{20}\right) + \frac{\frac{1}{1}}{15}$$

$$= \frac{1}{600} + 15 \times 60 + 15 = 1.5 + 15 = 16.5 \text{ minutes.}$$

Example 10.10. *In a restaurant, the customers are required to collect the coupons after making the payment at one counter, after which he moves to the second counter where he collects the snacks and then to the third counter, where he collects the cold drinks. At each counter he spends minutes on an average and this time of service at each counter is exponentially distributed. The arrival of customer is at the rate of 10 customers per hour and it follows Poisson's distribution. Determine*

(a) *Average time a customer spends waiting in the restaurant;*

(b) *Average time the customer is in queue.*

Solution. λ= 10 customer/our

NOTES

μ = Total service time for one customer

$= \frac{3}{2} \times 3 = \frac{9}{4}$ customers

$= \frac{4}{9} \times 60 = \frac{80}{3}$ hours.

(*a*) Average time a customer spends waiting in the restaurant $E_t = k + \frac{1}{2k} \times \frac{\lambda}{\mu(\mu - \lambda)}$

$\frac{4}{9} = 10 \times \frac{3}{80} \times \frac{80}{3} - 10 = \frac{1}{4} \times \frac{3}{50} = \frac{3}{200}$ minutes or $\frac{3}{200} \times 600 = 0.9$ minutes.

(b) Average time the customers in queue

$\frac{1}{\mu} = \frac{1}{\frac{80}{3}} = \frac{3}{80} \times 60 = \frac{9}{4}$ = minutes.

10.7 SUMMARY

- Queuing theory has been used for many real life applications to a great advantage. It is so because many problems of business and industry can be assumed/simulated to be arrival-departure or queuing problems.
- Queuing theory techniques, in particular, can help us to determine suitable number and type of service facilities to be provided to different types of customers.
- **Arrival Pattern:** It is the pattern of the arrival of Qa customer to be serviced.
- **Poisson's Distribution:** It is discrete probability distribution which is used to determine the number of customers in a particular time. It involves allotting probability of occurrence of the arrival of a customer.
- **Exponential Distribution:** This is based on the probability of completion of a service and is the most commonly used distribution in queuing theory.
- **Service Pattern:** We have seen that arrival pattern is random and poissons distribution can be used for use in queue model.
- **Channels:** A service system has a number of facilities positioned in a suitable manner.
- **Service Time:** Service time, i.e., the time taken by the customer when the facility is dedicated to it for serving depends upon the requirement of the customer and what needs to be done as assessed by the facility provider.
- **Idle Rate.** This is the rate at which the service facility remains unutilised and is lying idle.

10.8 REVIEW AND DISCUSSION QUESTIONS

1. What is a queue ? Give an example and explain the basic concept of queue.
2. Define a queue. Give a brief description of the type of queue discipline commonly faced.

3. (*a*) Explain the single channel and multi-channel queuing models.

(*b*) Draw a diagram showing the physical layout of a queuing system with a multi server, multi-channel service facility.

4. (*a*) Give some applications of queuing theory.

(*b*) State three applications of waiting line theory in business enterprises.

5. With respect to the queue system, explain the following :

(*i*) Input process, (*ii*) Queue discipline, (*iii*) capacity of the system, (*iv*) Holding time, (*v*) Balking and (*vi*) Jockeying.

6. Briefly explain the important characteristic of queuing system.

7. What do you understand by :

(*a*) (*i*) queue length, (*ii*) traffic intensity, (*iii*) the service channels ?

(*b*) (*i*) steady and transient state and (*ii*) utilization factor ?

8. Show that if the inter-arrival times are exponentially distributed, the number of arrivals in a period of time is a Poisson process and conversely.

9. Consider the pure birth process, where the system starts with K customers at $t = 0$. Derive the equation describing the system and then show that

$$\rho_n(t) = \frac{e^{-\lambda i}(\lambda t)^n}{(n-k)!};\ n = k, k+1, ...$$

10. Consider the pure birth process, where the number of departures in some time interval follows a Poisson distribution. Show that the line between successive departures is exponential.

11. If $\lambda\Delta t$ is the probability of a single arrival during a small interval of time Δt, and if the probability of more than one arrival is negligible, prove that the arrivals follows the Poisson's law.

12. (*a*) Derive Poisson's process assuming that the number of arrival, in non-overlapping intervals, are statistically independent and then apply the binomial distribution.

(*b*) What are the various queuing models available ?

13. Explain (*i*) Single queue, single server queuing system, and (*ii*) Single queue, multiple servers in series queues.

[Hint. GD indicates that discipline is general, i.e., it may be FCFS or LCFS or SIRO]

14. For a (M/M/1) : (∞/F/FO) queuing model, in the steady-state case, obtain expressions for the mean and variance of queue length in terms of relevant parameters : λ and μ.

15. For a (M/M/1) : (∞/F/FO) queuing model in the steady-state case, show that

(*a*) The expected number of units in the system and in the queue is given by

$$E(n) = \frac{\lambda}{(\mu-\lambda)} \text{ and } E(m) = \frac{\lambda^2}{\mu(\mu-\lambda)}$$

(*b*) (*i*) Expected waiting time of an arrival in the queue is $\frac{\rho}{\mu(1-\rho)}$.

(*ii*) Expected waiting time the customer spends in the system (including services) is

16. Define busy period of a queuing system. Obtain the busy period distribution for the simple (M/M/1) : (∞/FCFS) queue.

What is the condition that the busy period will terminate eventually ?

17. Derive the differential-differential equations for the queuing model (M/M/1) : (N/FCFS) and solve the same.

18. For a (M/M/1) : (N/FIFO) queuing model :

(*i*) find the expression for E(*n*),

(*ii*) derive the formula for P_n and E(*n*) when $\rho = 1$.

19. (*a*) For a (M/M/C) : (∞ / FCFS) queuing model, derive the expression for

(*i*) the steady state equation,

(*ii*) probability that a customer will not have to wait,

(*iii*) expected number of customer in the queue,

(*iv*) expected number of customers in the system,

(*v*) expected waiting time of a customers in the system,

(*vi*) probability of server to be idle.

(*b*) Giving clearly the assumptions, derive the steady state distribution of queue length in (M/M/K) queuing model.

20. For (M/M/C) ; (N/FCFS), derive the steady-state equations describing the situation fo N = C; then sow that the expression for P*n* is given by

$$Pn = \begin{cases} \dfrac{P_0\left(\frac{\lambda}{\mu}\right)^n}{n!}, & 0 \le n \le C \\ 0 \quad, & \text{otherwise} \end{cases}$$

where $$P_0^{-1} = \sum_{n=0}^{c} \frac{\left(\frac{\lambda}{\mu}\right)^n}{n!}.$$

21. For Erlang is distribution with parameters μ and *K* prove that the mode is at $\dfrac{(K-1)}{K\mu}$, the mean is $\dfrac{1}{\mu}$, and the variance is $\dfrac{1}{K\mu^2}$.

22. For (M/G/I) : (∞/FCFS) queuing model, derive the Pollaczek-Khintehine (P-K) formula for expected number of customers in the system.

23. Show hat for the special case of exponential service time with mean $\dfrac{1}{\mu}$, the results of (M/G/1) model reduce to those of the (M/M/1) model.

24. Under the standard queuing model nomenclature indicate what do you mean by the following:

M/M/S, D/M/1, M/G_k/S, G/G/S and E_k/GI/S

25. Write short notes on

(*i*) Cost-profit models in queuing theory.

(*ii*) Non-Poisson queues.

(*iii*) Information requirement, assumption and objectives of queuing models.

26. A foreign bank is considering opening a drive in window for customer service. Management estimates that customers will arrive for service at the rate of 12 per hour. The teller whom it is considering to staff the window and serve customers at the rate of one every three minutes. Assuming Poisson arrival and Exponent ice service find :

(*a*) Utilization of teller;

(*b*) Average number in the system;

(*c*) Average waiting time in the line;

(*d*) Average waiting time in the system.

UNIT 11: REPLACEMENT THEORY

NOTES

Structure

11.1 INTRODUCTION

Replacement of old plant and equipment and items of use like bulbs/tube-lights, refrigerators/ heating, tools/gadgets, etc, is a necessity. All these items are designed for performance up to the desired level for a particular time (years/hours) or particular number of operations. For example, when a refrigerator is given the warranty for 7 years the manufacturer knows that the design of the refrigerator is such that it will perform up to desired level of efficiency without breakdown for that period. Similarly, bulbs/tube-lights may have been designed for say 10,000 on-off operations. But all these need to be replaced after a particular period/number of operations. The equipment is generally replaced because of the following reasons:

(*i*) When the item/equipment fails and does not perform its function it is meant for.

(*ii*) Item/equipment has been in use for sometimes and is expected to fail soon.

(*iii*) The item/equipment in use has deteriorated in performance and needs expensive repairs *i.e.*, it has gone beyond the economic repair situations. The cost of maintenance and repair of equipment keeps increasing with the age of the equipment. When it becomes uneconomical to continue with an old equipment, it must be replaced by a new equipment.

(*iv*) Improved technology has given access to much better (convenient to use) and technically superior (using less power) products. This is the case of obsolescence. The equipment needs to be replaced not because it does not perform up to the standards it is designed for but because new equipment is capable of performance of much higher standards.

It should be understood that all replacement decisions involve high financial costs. The financial decisions of such nature will depend upon a large number of factors, like the cost of new equipment, value of scrap, availability of funds, cost of funds that have to be arranged, tax benefits, government policy, etc.

When making replacement decisions, the management has to make certain assumptions, these are :

(*i*) The quality of the output remains unchanged.

(*ii*) There is no change in the maintenance costs.

(*iii*) Equipments perform to the same standards.

Let us discuss some of the common replacement problems.

Replacement of items, which deteriorate with time without considering the change in money value

Most of the machinery and equipment having moving parts deteriorate in their performance with passage of time. The cost of maintenance and repair keeps increasing with passage of time and a stage may reach when it is more economical (in overall analysis) to replace the item with a new one. For example, a passenger car is bound to wear out with time and its repair and maintenance cost may go to such level that the owner has to replace it with a new one.

Let C = Capital cost of the item,

S (t) = Scrap value of the item after t years of use,

O (t) = Operating and maintenance cost of the equipment at time t,

n = number of years the item can be used,

TC (n) = Total cost of using the equipment for n years,

$$\text{TC}(n) = \text{C} - \text{S}(t) + \sum_{t=1}^{n} \text{O}(t)$$

$$\text{Average TC}(n) = \frac{1}{n\left[\text{C} - \text{S}(t)\sum_{t=1}^{n} \text{O}(t)\right]}$$

Time 't' in this case is a discrete variable.

In this case as long as the average TC (n) is minimum, the equipment can remain in use for that number of years. If average total cost keeps decreasing up to ith year and starts increasing from (i + 1)th year then ith year may be considered as most economic year for replacement of the equipment.

The concept of depreciation cost also must be understood here. As the years pass by, the cost of the equipment or items keeps decreasing. How much the cost keeps decreasing can be calculated by two methods commonly used, *i.e.,* straight line depreciation method and the diminishing value method.

Example 11.1. *A JCB excavator operator purchases the machine for Rs 1500000. The operating cost and the resale value of the machine is given below.*

Year	1	2	3	4	5	6	7	8
Operating Cost (in Rs)	30000	32000	36000	40000	45000	52000	60000	70000
Resale value (in lakhs of Rs.)	12	10	8	5	4.5	4	3	2

When should the machine be replaced ?

Solution.

C = 1500000

O (t) = Operating cost

S (t) = Resale value

t = Time

n = Number of years after when the machine is to be replaced.

Let us draw a table showing the various variables required to make decision. This is shown in the table below.

NOTES

Year	O (*t*) (in thousand of Rupees)	Cumulative O (*t*)	Resale value S (*t*) (in thousands of Rupees)	Depreciation C – S (*t*) (in thousands of Rupees)	Total cost TC (*n*) (in thousands of Rupees)	Average TC (n) (in thousands of Rupees)
1	30	30	1200	300	330	330
2	32	62	1000	500	562	281
3	36	98	800	700	798	266
4	40	138	500	1000	1138	284.5
5	45	183	450	1050	1233	246.6
6	52	235	400	1100	1335	222.5
7	60	295	300	1200	1495	213.6
8	70	365	200	1300	1665	208

In third year the minimum average cost is 266000 as shown in the table above. So, replacement should take place at the end of three year.

Example 11.2. *A new tempo costs Rs. 100000 and may be sold at the end of year at the following prices:*

Year	1	2	3	4	5	6
Selling Price (Rs.)	60000	45000	32000	22000	10000	2000

The corresponding annual operating costs are:

Year	1	2	3	4	5	6
Cost/Year (Rs.)	10000	12000	15000	20000	30000	45000

It is not only possible to sell the tempo after use but also to buy a second hand tempo. It may be cheaper to do so than to buy a new tempo.

Age of tempo	0	1	2	3	4	5
Purchase Price (Rs.)	100000	60000	45000	33000	20000	10000

What is the age to buy and to sell so as to minimize average annual cost ?

Solution. Cost of new tempo = Rs. 100000

Let us find out the average cost per year of the new tempo.

Year of Service (1)	Operating Cost O (t) (2)	Cumulative Operating Cost O (*t*) (3)	Resale Value S (t) (4)	Depreciation C – S (*t*) (5)	Total Cost TC (*n*) (6) = (3) + (5)	Average TC (*n*) (7) = (6) ÷ (1)
1	10000	10000	60000	40000	50000	50000
2	12000	22000	45000	55000	77000	38500
3	15000	37000	32000	68000	105000	35000
4	20000	57000	22000	78000	135000	33750
5	30000	87000	10000	90000	177000	35400
6	45000	132000	2000	98000	230000	35000

Average cost is minimum at the end of fourth year; hence the new tempo should be replaced after 4 years.

Let us now find out the average total cost of second hand tempo.

Year of Service (1)	**Operating Cost O (*t*) (2)**	**Cumulative Operating Cost O (*t*) (3)**	**Resale Value S (t) (4)**	**Depreciation C – S (*t*) (5)**	**Total Cost TC (*n*) (6) = (3) + (5)**	**Average TC (*n*) (7) = (6) ÷ (1)**
0			100000	100000	100000	
1	10000	10000	60000	40000	50000	50000
2	12000	22000	45000	55000	77000	38500
3	15000	37000	32000	68000	105000	35000
4	20000	57000	20000	78000	137000	33750
5	30000	87000	10000	90000	177000	35400

The tempo may be replaced by second hand tempo at the end of third year and the owner can save Rs. (35000–34666), *i.e.*, Rs. 334 instead of buying a new one.

Example 11.3. (*a*) *Machine A costs Rs. 9000. Annual operating costs are Rs. 200 for the first year and then increases Rs. 2000 every year. Determine the best age at which to replace the machine. If the optimum replacement policy is followed, what will be the average yearly cost of owning and operating the machine?*

(*b*) *Machine B costs Rs. 10000. Annual operating cost are Rs. 400 for the first year and then increases by Rs 800 every year. You now have a machine of type A which is one year old. Should you replace it with B, if so, when ?*

Solution. (*a*) Let us assume that there is no scrap value of the machine Average total cost can be computed as

Year (*n*)	Operating Cost O (*t*)	Cumulative Operating Cost Σ O (*t*)	Depreciation| C – S (*t*)	Total Cost	Average Cost
1	200	200	9000	9200	9200
2	2200	2400	for all years	11400	5700
3	4200	6600		15600	5200
4	6200	12800		21800	5450
5	8200	21000		30000	6000

It can be seen that the best age for replacement is third year.

(b) For machine B, the average cost can be calculated as follows:

Year (*n*)	Operating Cost O (*t*)	Cumulative Operating Cost Σ O (*t*)	Depreciation| C – S (*t*)	Total Cost	Average Cost
1	400	400	10000	10400	10400
2	1200	1600	for all year	11600	5800
3	2000	3600		13600	4533
4	2800	6400		16400	4100
5	3600	10000		20000	4000
6	4400	14400		24400	4066

Since the minimum average cost for machine B is lower than for machine A, machine B should be replaced by machine A. Minimum average cost is (Rs. 4000), it should be replaced when it exceeds Rs. 4000. In case of one year old machine Rs. 2200/- will be spent next year and Rs. 4200 the following year. We should keep machine A for one year.

Replacement policy of an equipment/item whose operating cost increases with time and money value also changes with time

In previous examples, we assumed that the money value does not change and remains constant but it is well-known that as the equipment deteriorates and operating costs keep increasing, the money value keeps decreasing with time. Hence we must calculate the *Net Present Value* (NPV) of the money to be spent a few years hence. Otherwise the resale value, the operating costs, which are to take place in future, will not be realistic and management will not be able to take optimal decisions.

Let C = Initial cost of item/equipment

O C = Operating cost

R = Rate of interest

A rupee invested at present will be equivalent to $(1 + r)$ a year after $(1 + r)^2$ two years hence and $(1 + r)^n$ in n years time. It means that making a payment of one rupee after n years is equivalent to paying $(1 + r)^n$ now. The quantity $(1 + r)^{-n}$ is called the *present worth* or *present value* of one rupee spent n years from now.

Present value of a rupee $V = (1 + r)^{-1} = 1/1 + r$ is called *discount rate* and is always less than 1.

Then, year wise present value of expenditure in future years can be calculated as

$$\text{Present value } (n) = (c + oc_1) + oc_2\, v + oc_3\, v^2 + \ldots\ldots\ldots + oc_n\, v^{n-1} + (c + oc_1)\, v^n + oc_2\, v^{n+1} + oc_3\, v^{n+2} + \ldots\ldots\ldots + oc_n\, v^{2n-1} + (c + oc_1)\, v^{2n} + oc_2\, v^{2n+1} + \ldots\ldots\ldots + oc_n\, v^{3n-1}$$

Steps Involved in Calculation of Replacement Policy When Time Value Changes

Step I. Find out the present value factor at the given rate and multiply it with the operating/ maintenance cost of the equipment/items for different years.

Step II. Work out the total cost by adding the cumulative present value to the original cost for all the years.

Step III. Cumulate the discount factors.

Step IV. Divide the total cost by corresponding value of the cumulated discount factor for every year.

Step V. Find out the value of last column that exceeds the total cost. Equipment/item will be replaced in the latest year.

These steps will be explained with the help of an example.

Example 11.4. *The yearly cost of two machines X and Y, when money value is neglected is shown below. Find which machine is more economical if money value is 10% per year.*

Year	1	2	3
Machine X (Rs.)	2400	1600	1800
Machine Y (Rs.)	3200	800	1800

Solution. It may be seen that the total cost for each machine X and Y is Rs. 5800 (2400 + 1600 + 1800) or (3200 + 800 + 1800). When the money value is not discounted the machines are equally good, total cost wise, when money value is not changed with time, with money value 10% per year, the discount rate, it changes as follows :

$$V = \frac{1}{1+r} = \frac{1}{1+0.10} = \frac{1}{1.1} = 0.9091$$

Discounted costs are obtained by multiplying the original costs with 0·9091 after one year. Total costs of machines X and Y are calculated as shown below.

NOTES

The total cost of machine X is less than that of machine Y, machine X is more economical.

Example 11.5. *A manufacturer is offered two machines X and Y. Machine X is priced at Rs. 10000 with running cost of Rs. 1000 for first four years and increasing by Rs. 400 in fifth year and subsequent years. Machine Y which has the same capacity and performance as X costs Rs. 8000 but has maintenance cost of Rs. 1200 per year for first five years increasing by Rs. 400 in the sixth and subsequent years. If cost of money is 10% per year, which is a more economical machine ? Assume running cost is incurred at the beginning of the year.*

Solution. $PV = \frac{1}{1+0{\cdot}10} = 0.909$

Machine X, C = 1000.

Year	OC	PV factors	PV of OC	C + Cumulative PV of OC	Cumulative PV Factor	Weighted Average Cost
1	1000	1·00	1000	11000	1·00	11000
2	1000	0·909	909	11909	1·909	5984·5
3	1000	0·826	826	12735	2·735	4656·30
4	1000	0·751	751	13486	3·486	3868·6
5	1400	0·683	956	14442	4·169	3464
6	1800	0·621	1116	15558	4·790	3248
7	2200	0·564	1240	16798	5·355	3137
8	2600	0·513	1326	18124	5·868	30886
9	3000	0·466	1398	19532	6·334	3084
10	3400	0·424	1429	21951	6·759	3247

Machine Y, C = 800.

Year	OC	PV factors	PV of OC	C + Cumulative PV of OC	Cumulative PV Factor	Weighted Average Cost
1	1200	1 × 00	1200	9200	1 × 00	9200
2	1200	0 × 909	1090 × 8	10290 × 8	1 × 909	5390 × 67
3	1200	0 × 826	991 × 2	11282	2 × 735	4125
4	1200	0 × 751	901 × 2	12183 × 2	3 × 486	3494 × 8
5	1200	0 × 683	819 × 6	13002 × 8	4 × 169	3119
6	1600	0 × 621	993 × 6	13996 × 4	4 × 790	2922
7	2000	0 × 564	1128	15124 × 4	5 × 355	2824 × 35
8	2400	0 × 513	1231 × 8	16355 × 6	5 × 868	2787 × 25
9	2800	0 × 466	1304 × 8	17660 × 4	6 × 334	2788 × 2
10	3200	0 × 424	1356 × 8	19017 × 2	6×759	2813 × 6

It can be seen that weighted average cost of machine X is minimum, *i.e.,* Rs. 3084 in ninth year. Where as the weighted average cost of machine Y is minimum in 8th year, *i.e.,* 2787.25 so it is advisable to purchase machine Y.

NOTES

11.2 REPLACEMENT POLICY FOR EQUIPMENT/POLICY THAT BREAKS DOWN/FAILS SUDDENLY

As an equipment or item, which is made of a number of components ages with time, it deteriorates in its functional efficiency and the performance standard are reduced. However, in real life situation there are many such items whose performance does not deteriorate with time but fail suddenly without any warning. This can cause immense damage to the system or equipment and inconvenience to the user. When the item deteriorates with time, one is expecting reduced performance but other items, which may fail without being expected to stop performing, can create a lot of problems. A minor component in an electronic device or equipment like TV, fridge or washing machine, costs very little and may be replaced in no time but the entire equipment fails suddenly if the component fails. Hence the cost of failure in terms of the damage to the equipment and the inconvenience to the user is much more than the cost of the item.

If it is possible to know exactly the life of the component, it is possible to predict that the component and hence equipment is likely to fail after performance of so many hours or miles, etc. This is the concept of preventive maintenance and preventive replacement. If the equipment is inspected at laid down intervals to know its conditions, it may not be possible to expect the failure of the item. The cost of failure must be brought down to minimum, preventive maintenance is cheap but avoids lots of problems. In many cases, it may not be possible to know the time of failure by direct inspection. In such cases the probability of failure can be determined from the past experience. Finding the Mean Time Between Failure (MTBF) of the equipment in past is one good way of finding this probability. It is possible by using the probabilities to find the number of items surviving up to certain time period or the number of items failing in a particular time period.

In situations, when equipment/item fails without any notice, two types of situations arise.

(*a*) Individual Replacement Policy. In this case an item is replaced immediately when it fails.

(*b*) Group Replacement Policy. In this policy all the items are replaced irrespective of the fact whether the items have failed or not, of course, any item failing before the time fixed for group replacement is also replaced.

Individual Replacement Policy

In this policy, a particular time '*t*' is fixed to replace the item whether it has failed or not. It can be done when one knows that an item has been in service for a particular period of time and has been used for that time period. In case of moving parts like bearings, this policy is very useful to know when the bearing should be replaced whether it fails or not. Failure of a bearing can cause a lot of damage to the equipment in which it is fitted and the cost of repairing the equipment is much more than the cost of bearing if it had been replaced well in time. If it is possible to find out the optimum service life '*t*' the sudden failure and hence loss to the equipment and production loss, etc, can be avoided. However, when we replace items on a fixed interval of preventive maintenance period certain items may be left with residual useful life which goes waste. Such items could still perform for another period of time (not known) and so the utility of items has been reduced. Consider the case of a city corporation wanting to replace its street lights.

If individual replacement policy is adopted then replacement can be done simultaneously at every point of failure. If group replacement policy is adopted then many lights with residual life will be replaced incurring unnecessary costs.

Analysis of the cost of replacement in case of items/equipments that fail without warning is similar to finding out the probability of human deaths or finding out the liability of claims of Life Insurance Company on the death of a policy holder.

The probability of failure or survival at different times can be found out by using mortality tables or life tables.

The problem of human births and deaths as also individual problems where death is equivalent to failure and birth is equivalent to replacement can also be studied as part of the replacement policy. For solving such problems, we make the following assumptions:

(a) All deaths or part failures are immediately replaced by births or part replacements and

(b) There are no other, except the ones under consideration, entries or exits.

Let us find out the rate of deaths that occur during a particular time period assuming that each item in a system fails just before a particular time period. The aim is to find out the optimum period of time during which an item can be replaced so that the costs incurred are minimum. Mortality or life tables are used to find out the probability destination of lifespan of items in the system.

Let $f(t)$ – number of items surviving at time $(t-1)$ n = Total number of items with system under consideration. The probability of failure of items between 't' and $(t-1)$ can be foud out by

$$P = \left(\frac{(t-1)-f(t)}{n}\right)$$

Replacement Policy

Let the service life time of an item be T and n = number of items in a system which need to be replaced whenever any of these fails or reaches T.

F (t) = number of items surviving at T

F¢(t) = 1 – f (t) number of items that have failed

O (t) = Total operating time

Cf = Cost of replacement after failure of item

CPM = Cost of preventive maintenance

Cost of replacement after failure of service time T = $n \times f'(t)$ X C_f

Also cost of replacement for item replaced before failure = $n\,[1 - f'(T)]\,C_{pm}$

$$= n + f'(T)\,cf + n\,[1 - f'(T)\,C_{pm}$$

Hence we can replace an item when the total replacement cost given above is minimum where

$$O(t) = f(t)\,dt$$

Group Replacement Policy

Under this policy, all items are replaced at a fixed interval 't' irrespective of the fact they have failed or not and at the same time keep replacing the items as and when they fail. This policy is applicable to a case where a large number of identical low cost items which are more and more likely to fail at a time. In such cases, *i.e.*, like the case of replacement of street lights, bulbs, it may be economical to replace all items at fixed intervals.

Let n = total number of items in the system

$N(t)$ = number of items that fail during time t

NOTES

$C(t) =$ Cost of group replacement after time t

$C(t)/t =$ average cost per unit time

$Cg =$ Cost of group replacement

$Cf =$ Cost of replacing one item on failure

$C(t) = nCg + Cf(n_1 + n_2 + \dots\dots + n_{t-1})$

$F(t) =$ Average cost per unit time $= C(t)/t = nCg + Cf(n_1 + n_2 + \dots\dots + n_{t-1})/t$

We have to minimize average cost per unit time, so optimum group replacement time would be that period which minimize this time.

It can be concluded that the best group replacement policy is that which makes replacement at the end of 't'th period if the cost of individual replacement for the same period is more than the average cost per unit time.

Example 11.5. *The following mortality rates have been observed for certain typ of light bulbs:*

End of week	1	2	3	4	5
Percentage Failing	10	20	50	70	100

There are 1000 bulbs in use and it costs Rs. 10 to replace an individual bulb which has burnt out. If all the bulbs are replaced simultaneously, it would cost Rs. 5 per bulb. It is proposed to replace all the bulbs at fixed intervals whether they have fixed or not and to continue replacing fused bulbs as and when they fail. At what intervals should all the bulbs be replaced so that the proposal is economical ?

Solution. Average life of a bulb in weeks = Probability of failure at the end of week × number of bulbs

$= (1 \times 10/100 + 2 \times 10/100 + 3 \times 30/100 + 4 \times 20/100 + 5 \times 30/100)$

$= 0.10 + 0.2 + 0.9 + 0.8 + 1.5 = 3.5$

Average number of replacement per week $= \dfrac{\text{number of bulbs}}{\text{average life}} = \dfrac{1000}{3.5} = 285$

Cost per week @ Rs 10 per bulb = 285 × 10 = Rs. 2850

Let n_1, n_2, n_3, n_4 and n_5 be the number of bulbs being replaced at the end of first,second, third, fourth and fifth week respectively then

$n_1 =$ number of bulbs in the beginning of the first week × probability of the bulbs failing during first week = 1000 × 10/100 = 100

$n_2 =$ (number of bulbs in the beginning × probability of the bulbs failing during second week) + number of bulbs replaced in first week × probability of these replaced bulbs failing in second week.

$= 1000 \times (20 - 10)/100 + 100 \times 10/100 = 100 + 10 = 110$

$n_3 =$ (number of bulbs in the beginning × probability of the bulbs failing during third week) + number of bulbs being replaced in first week × probability of these replaced bulbs failing in second week) + number of bulbs being replaced in second week × probability of those failing in third week)

$= 100 \times (50 - 20)/100 + 100 \times (20 - 10)/100 + 110 + 10/100 = 300 + 10 + 11 = 321$

$n_4 = 1000 \times (70 - 50)/100 + 100 \times (50 - 20)/100 + 110 + 20 - 10/100 + 321 \times 10/100$

$= 200 + 30 + 11 + 32 = 273$

$n_5 = 100 \times 30/100 + 100 \times 20/100 + 110 \times 30/100 + 321 \times 10/100 + 273 \times 10/100$

$= 300 + 20 + 33 + 32 + 28 = 413$

The economics of individual or group replacement can be worked out as shown in the table below.

End of week	No. of bulbs failing	Cumulative No. of failed bulbs	Cost of individual replacement	Cost of group replacement	Total cost	Average Total Cost
1	100	100	1000	5000	6000	6000
2	110	220	2200	5000	7200	3600
3	321	541	5410	5000	10410	3470
4	273	814	8140	5000	13140	3285
5	413	1227	12270	5000	17270	3454

Individual replacement cost was worked out to be Rs. 2850. Minimum average cost per week corresponding to 4th week is Rs. 3285, it is more than individual replacement cost. So it will be economical to follow individual replacement policy.

Example 11.6. *The computer system has a large number of transistors. These are subject to a mortality rte given below :*

Period	Age of failure in hours	Probability of failure
1	0 – 400	0.05
2	400 – 800	0.15
3	800 – 1200	0.35
4	1200 – 1600	0.45

If the transistors are replaced individually the cost per transistor is Rs. 20. But if it can be done as a group at a specific interval determined by the preventive maintenance policy of the user, then the cost per transistor comes down to Rs. 10. Should the transistor be replaced individually or as a group ?

Solution. Let us assume that a block of 400 hours is the one period and total number of transistors in the system are 1600.

Find out the average failure of transistors

Average failure = Number of transistors /Average mean life

$= 1600 / (0.05 \times 1 + 0.15 \times 2 + 0.35 \times 3 + 4 \times 0.45)$

$= 1600 / (0.05 + 0.30 + 1.05 + 1.80)$

$= 1600/3.2 = 500$

If cost of individual replacement policy is adopted, Cost = Rs. 500 × 20 = Rs. 10000, now we must find out the failure of transistors per period of block of 400 hours.

$= 1600/3.2 = 500$

$n_1 = n_0 p_1 = 1600 \times 0.05 = 80$

$n_2 = n_0 p_2 + n_1 p_1 = 1600 \times 0.15 + 80 \times 0.05 = 240 + 4 = 244$

NOTES

$n_3 = n_3 = n_0 p_3 + n_1 p_2 + n_2 p_1 = 1600 \times 0.35 + 80 \times 0.15 + 244 \times 0.05$
$= 560 + 12 + 12.2 = 585$

$n_4 = n_0 p_4 + n_1 p_3 + n_2 p_2 + n_3 p_1 = 1600 \times 0.45 + 80 \times 0.35 + 244 \times 0.15 + 585 \times 0.05$
$= 720 + 28 + 36.6 + 29.25 = 814$

Now, average cost of group replacement must be found.

Period 400 hours block	Failure of ICs during month	Cumulative failure	Individual replacement cost @ 20	Group replacement cost @ 10	Total cost	Average Total Cost
1	80	80	1600	1600 × 10 = 16000	17600	17600
2	244	324	6480		222480	11240
3	585	909	18180		34180	11393
4	814	1723	34460		50460	12615

The minimum cost of group replacement is Rs. 11240 for an interval of 400 hours. Individual replacement is optimal policy since the cost is Rs. 10000, which is less than the group replacement cost.

Manpower replacement policy (Staffing policy)

All organizations face the problem of initial recruitment and filling up of vacancies caused by promotions, transfer, employee quitting their jobs or retirement and deaths. The principle of replacement used in industry for replacement of parts, etc, can also be used for recruitment and promotion policies, which are laid down as personnel policy of an organization. The assumption made in such case is that the destination of manpower is already decided. Few examples will illustrate this point.

Example 11.7. *An army unit requires 200 men, 20 Junior Commissioned Officers (JCOs) and 10 officers. Men are recruited at the age of 18 and JCOs and officers are selected out of these. If they continue in service, they retire at the age of 40. At present there are 800 jawans and every year 20 of them retire. How many jawans should be recruited every year and at what age promotions should take place ?*

Solution. If 800 jawans had been recruited for the past 22 years (age of recruitment 40 years – age of entry 18 years), the total number of them serving up to age of 39 years = 20 × 22 = 440

Total number of jawans required = 200 + 20 + 10 = 230

Total number of jawans to be recruited every year in order to maintain a strength of 230

= 800/440 × 230 = 418

Let a jawan be promoted at age of X, then up to X – 1 year, number of jawans recruited is 200 out of 230. Hence out of 800, jawans required = 200/230 × 800 = 696.

696 will be available up to 5 years as 20 retire every year and (800 – 20 × 5) = 700. Hence promotion of jawans is due in sixth year.

Out of 230 jawans required, 20 are JCOs, therefore if recruited 800, number of JCOs = 20/230 × 800 = 70 approximately.

In a recruitment of 800, total number of men and JCOs = 697 + 70 = 766

Number of officers required = 800 – 766 = 34

This number will be available in 20 years of service, so promotion of JCOs to officers is due in 21 year of service.

NOTES

Example 11.8. *College X plans to raise the strength of its faculty to 150 and then keep it at that level. The wastage of faculty due to retirement, quitting, deaths, etc, based on the length of service of the faculty member s as given below.*

Block years	1	2	3	4	5	6	7	8	9	10
% of teachers	0	5	10	15	20	30	35			
Those who live up to end of year	0	5	35	60	65	70	85	100		

(*i*) Find the number of faculty members to be recruited every year.

(*ii*) If there are 10 posts of Head of Departments (HODs) for which length of service is the only criterion of promotion, what is the average length of service after which a new faculty member should expect promotion ?

Solution. Let us assume that the recruitment per year is 100. These 100 teachers join initially in the block of 0 – 5 years, will serve for 35 years and will become 0 in 7th block of 5 years, *i.e.*, at the service of 35 years. Those 100 who join between the block of year 5 – 10 will serve for 30 years and become 15, the third set of teachers will become 30 after 25 years of service and soon.

Year	**No. of faculty members**
0	100
5	95
10	65
15	40
20	35
25	30
30	15
35	0

Hence, if 100 faculty members are required every year, the total number of staff members after 35 years (7 block of 5 years) of service = 380

To maintain staff strength of 150, the number to be recruited every year = 100 /380 × 150 = 40

If the college recruits 40 every year, then they want 10 as HODs. Hence if the college recruits 100 every year then they will need HODs = 10/40 × 100 = 25.

It can be seen from the above table that 0 + 15 + 30 ≥ 25, *i.e.*, the promotion of the newly recruited faculty member as HODs can be done after 20ars of service.

11.3 SUMMARY

- Replacement of old plant and equipment and items of use like bulbs/tube-lights, refrigerators/heating, tools/gadgets, etc, is a necessity. All these items are designed for performance up to the desired level for a particular time (years/hours) or particular number of operations.
- When making replacement decisions, the management has to make certain assumptions, these are :

(*i*) The quality of the output remains unchanged.

(*ii*) There is no change in the maintenance costs.

(*iii*) Equipments perform to the same standards.

NOTES

- Most of the machinery and equipment having moving parts deteriorate in their performance with passage of time. The cost of maintenance and repair keeps increasing with passage of time and a stage may reach when it is more economical (in overall analysis) to replace the item with a new one.

11.4 REVIEW AND DISCUSSION QUESTIONS

1. What is replacement problem ? When does it arise ?

2. Describe various types of replacement situations.

3. Enumerate various replacement problems.

4. What are the situations which make the replacement of items necessary ?

5. Give a brief account of situations of which the replacement problems arise. What does the theory of replacement establish ?

6. Discuss in brief replacement procedure for the items that deteriorate with time.

7. The cost of maintenance of a machine is given as a function increasing with time and its scrap value is constant. Show that the average annual cost will be minimized by replacing the machine when the average cost to date becomes equal to the current maintenance cost.

8. Discuss the replacement problem where items are such that maintenance costs increase with time and the value of money also changes with time.

9. Find the optimum replacement policy which minimizes the total of all future discounted costs for an equipment which costs Rs. A and which needs maintenance costs of Rs. C_1 , C_2 , …, C_n etc.($C_{n+1} > C_n$) during the first year, second year etc., and further D is the depreciation value per unit of money during a year.

10. State some of the simple replacement policies.

11. Construct the cost equation reflecting the discounted value of all future costs for a policy of replacing equipment after every *n* periods. Hence establish the following :

(*i*) Replace if the next period cost is greater than the weighted average of previous costs.

(*ii*) Do not replace if the next period's cost is less than the weighted average of previous costs.

12. What is "group replacement"? Give an example.

13. Write a short note on group replacement and individual replacement policies.

14. The cost per item for the individual replacement is C_1 and the cost per item of group replacement is C_2 . If only individual replacement is more economical than the group replacement along with the individual, find relation between C_1 and C_2 .

15. A truck has been purchased at a cost of Rs. 160000. The value of the truck is depreciated in the first three years by Rs. 20000 each year and Rs. 16000 per year thereafter. Its maintenance and operating costs for the first three years are Rs. 16000, Rs. 18000 and Rs. 20000 in that order and increase by Rs. 4000 every year. Assuming an interest rate of 10% find the economic life of the truck.

16. A manual stamper currently valued at Rs. 10000 is expected to last 2 years and costs Rs. 4000 per year to operate. An automatic stamper which can be purchased for Rs. 3000 will last 4 years and can be operated at an annual cost of Rs. 3000. If money carries the rate of interest 10% per annum, determine which stamper should be purchased.

17. The cost of a new machine is Rs. 5000. The maintenance cost of nth year is given by

$R_n = 500\ (n - 1);\ n = 1, 2, \ldots\ldots$.

Suppose that the discount rate per year is 0.05. After how many years will it be economical to replace the machine by new one ?

18. A machine costs Rs. 10000 operating costs are Rs. 500 per year for the first five years. Operating costs increase by Rs. 100 per year in the sixth and succeeding years. Assuming a 10 per cent discount rate of money per year, find the optimum length of time to hold the machine before it is replaced. State clearly the assumptions made.

19. An individual is planning to purchase a car. A new car will cost Rs. 120000. The reasale value of the car at the end of the year is 85% of the previous year value. Maintenance and operation costs during the first year are Rs. 20000 and they increase by 15% every year. The minimum resale value of the car can be Rs. 40000.

(*i*) When should the car be replaced to minimum average annual cost (ignore interest) ?

(*ii*) If interest of 12% is assumed, when should the car be replaced ?

20. A large computer installation contains 2000 components of identical nature which are subject to failure as per probability distribution given below :

Weekend :	1	2	3	4	5
Percentage failure to date:	10	25	50	80	100

Components which fail have to be replaced for efficient functioning of the system. If they are replaced as an when failure occur, the cost of replacement per unit is Rs. 3. Alternatively, if all components are replaced in one lot at periodical intervals and individually replaced only as such failures occur between group replacement, the cost of component replaced is Re. 1.

(*a*) Access which policy of replacement would be economical.

(*b*) If group replacement is economical at current costs, then assess at what cost of individual replacement would group replacement be uneconomical.

(*c*) How high can the cost per unit in group replacement be to make a preference for individual replacement policy ?

21. Let $p\ (t)$ be the probability that a machine in a group of 30 machines would breakdown in period t. The cost of repairing a broken machine is Rs. 200. Preventive maintenance is performed by servicing all the 30 machines at the end of T unit of time. Preventive maintenance cost is Rs. 15 per machine. Find optimum T which will minimize the expected total cost per period of servicing, iven that

$$p(t) = \begin{cases} 0.03 & \text{for } t = 1 \\ p(t-1) + 0.01 & \text{for } t = 2, 3, \ldots, 10 \\ 0.13 & \text{for } t = 11, 12, 13, \ldots \end{cases}$$

What is the optimum replacement plan ?

NOTES

22. A manufacturer wants to replace a machine. The purchase price of the machine is Rs. 10000. Following other detail are available:

Year	Maintenance (Rs.)	Resale Price (Rs.)
1	1200	6000
2	1300	3000
3	1500	2000
4	2000	1000
5	2200	800
6	3000	500
7	3200	400
8	3800	300

Suggest in which year the machine may be replaced, if the supplier of the machine is prepared to prove 3 years insitu maintenance free of cost.

UNIT 12: PROJECT MANAGEMENT PERT AND CPM

Structure

12.1 INTRODUCTION

Programming Evaluation and Review Technique (PERT) and Critical Path Method (CPM) are two techniques used in project management. Project management is necessary to ensure that a project is completed within the stipulated budget, within the allocated time and perform to satisfaction.

PERT was developed by US Navy in 1958 for managing its Polaris Missile Project. It is very useful device for planning time and resources of a project. Polaris Missile project involved 3000 separate contracting organizations and was regarded as the most complex project experience till that time.

Parallel efforts, at almost the same time, were undertaken by Du Pont Company, which developed Critical Path Method (CPM) to plan and control the maintenance of chemical plants. These methods were subsequently widely used by Du Pont for many engineering functions.

12.2 PROJECT MANAGEMENT

Definition of terms commonly used in PERT and CPM

Activity

Activity is the smallest unit of productive efforts to be planned, scheduled and controlled. It is an identifiable part of the project, which consumes time and resources. In fact, a project is a combination of interrelated activities, which must be performed in a certain order for its completion. The project is divided into different activities by the work breakdown into smaller work contents. In network (arrow diagram) an activity is represented by an arrow, the tail that represents the start and the head, the finish of the activity. The length, shape and direction of the arrow have no relation to the size of the activiy.

NOTES

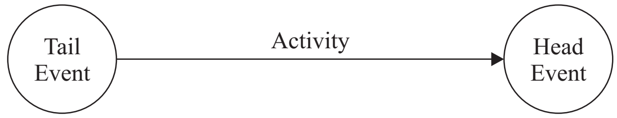


Fig. 12.1

Event

An event is an instant of time at which an activity starts and finishes. An event is represented by a node, *i.e.,* O. The beginning of an activity is *Tail Event* and finishing of an event is *Head Event*.

Path

An unbroken chain of activity arrows connecting the initial event to some other event is called a path.

Predecessor Activity

This is an activity that must be completed immediately before the start of another activity.

Successor Activity

Activity, which cannot be started until one or more activities are completed but immediately succeeds them is called successor activity of a project.

Dummy Activity

As seen in the definition of activities, all activities take some time and resources. A dummy activity is the one which is introduced in the network for communication when two or more activities have the same head and tail events. It means that two or more activities share the same start and finish nodes simultaneously. A dummy activity takes no time and requires no resources. It is shown as a dotted line in figure 15.2. Figure 15.3 shows wrong represetation

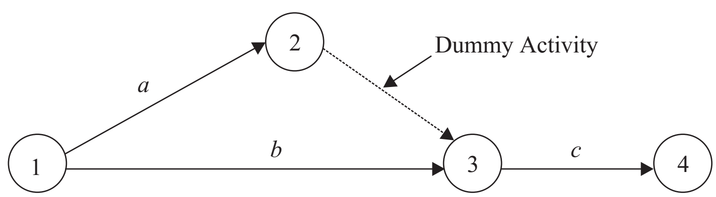


Fig. 12.2. *Correct Representation*

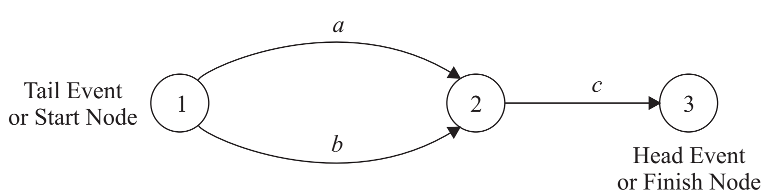
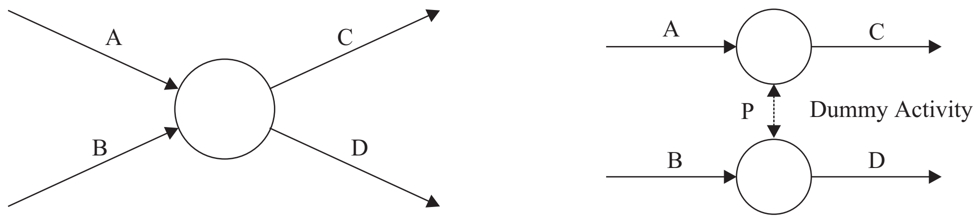


Fig. 12.3. *Wrong Representation*

Let us assume that the start of activity C depends upon the completion of activity A and B and the start of activity D depends only on the finish of activity. For this situation, we draw wrong and right representation in figure 15.4 to understand the introduction of dummy activity in etwork.

NOTES



(a) Wrong Representation (b) Correct Representation

Fig. 12.4

12.3 NETWORK (ARROW DIAGRAM)

A network is the graphical representation of logically and sequentially connected arrows representing activities and nodes representing events of a project.

Looping

Sometimes, due to errors in network logic, a situation of looping or cycling error occurs in which no activity can be completed as all the activities of the network are interlinked. In such situations, there is need to re-examine the network logic and redraw the network. To understand looping, see Figre 15.5.

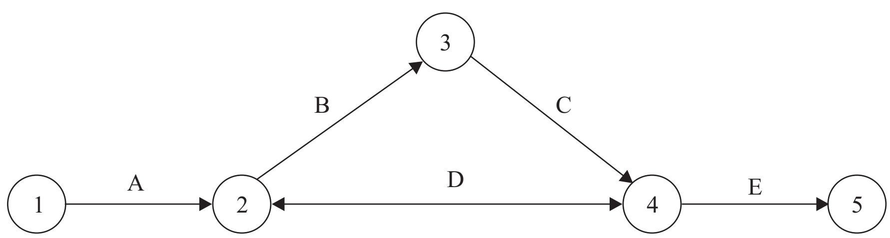


Fig. 12.5

Activity B cannot start until activity D is completed and activity D depends on the completion of activity C but C is dependent on the completion of activity B. Thus activities B, C and D form a loop and the network cannot proceed. Such condition can be avoided by checking the precedence relationship of the activities and numbering them in a logical sequence.

Dangling

In a network all activities except the final activity has a successor activity. A situation may occur when an activity other than the final activity, does not have a successor activity. The situation is shown in FIgure 9.6.

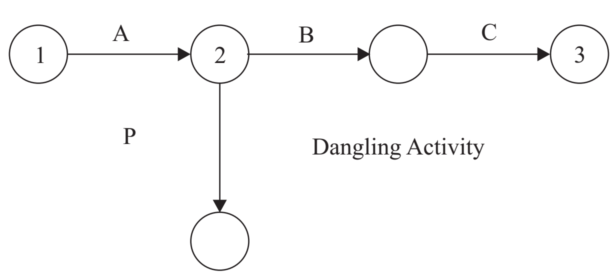


Fig. 12.6

In must be remembered that except the first node and last node, all nodes must have at least one activity entering it and one activity leaving it.

NOTES

Construction of Networks

Construction of a network is a simple procedure of putting all the events and activities in a logical and sequential manner to meet the requirement of a particular project/ problem. Difficulty occurs only when the basic rules are ignored. The following steps are helpful in constructing the network:

(*a*) Divide the project into activities by following the procedure of Work Breakdown Structure (WBS).

(*b*) Decide the start event, and the end event of project for all the activities. This is called establishing the precedence order and is the most important part of drawing the network.

(*c*) The activities decided by the precedence order are put in a logical sequence by using the graphical representation notations. Logical sequence can be decided by asking the following questions :

(*i*) What are the activities that must be completed before the start of a particular activity ? (Predecessor Activities)

(*ii*) What activities must follow the activity already drawn ? (Successor Activities)

(*iii*) Are there any activities which must be performed simultaneously with a particular activity ?

Rules to construct a network

1. Activities are represented by arrows ⟶ and events are represented by circles O.
2. Each activity is represented by one and only one arrow. The tail of the arrow represents the start and head the end of the activity.
3. Each activity must start and end in a node.
4. Arrow representing activities must be kept straight and should not be shown curved or bent.
5. Angles between arrows should be as larges as possible to make the activities clearly distinguishable from each other.
6. Arrows should not cross each other.
7. Event Number 1 represents the start of the project. There will be no activities (arrows) entering this node.
8. All events (nodes) should be numbered in an ascending order.
9. No events numbers can be repeated.
10. Dangling is not permitted.
11. Dummy activities also must follow the above rules, even though they do not consume any resource or time.

Numbering of events

For numbering of the events, Fulkerson's Rule is very helpful.

(a) Initial or start event, having no preceeding event is numbered 1.

(b) Numbering of other events is done from left to right or from top to bottom as 2, 3, 4, etc.

NOTES

(c) The events, which has been numbered are ignored or deleted. This will result in new initial events; these must be numbered in ascending order.

(d) Continue numbering all the events till we reach the last event out of which no activity (arrow) will emerge. It will be allotted the highest number, as it is the end event.

The numbering of activities is illustrated with the hel of Figure 15.7 :

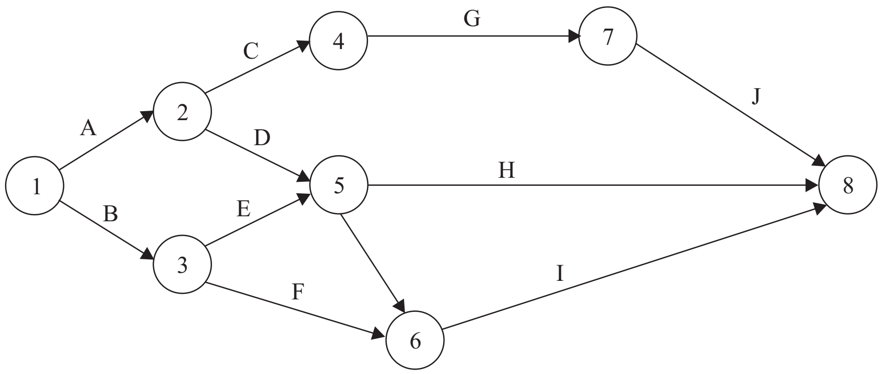


Fig. 12.7

Skipping of event numbers

In large projects in which the activities run into hundreds, it is not always possible to list all the activities at the initial stage and some additional activities may have to be added as the project progresses. Hence, while numbering the events continuously as 1, 2, 3, 4...and so on, the events are numbered in gaps of 5's or 10's so that other events can be inserted without causing any inconvenience to the logic of the network. The first event may be numbered 5 and subsequent events may be numbered as 10, 15, 20 and so on.

Example 12.1. *Let us use a simple example to illustrate the procedure we have just learnt. Listed below is the precedence chart showing the activities, their precedence (sequence), etc., for the project, ' Launching a new product' Sequencing is very important part of the construction of a network. The precedence given below must be carefully understood, as this example will be used to draw the network t a later stage.*

Activity	*Description*	**Immediate predeces-sor activity**	**Time (weeks)**
A	*Arranging a sales office*	–	*6*
B	*Hiring sales persons*	*A*	*4*
C	*Training sales persons*	*B*	*7*
D	*Selecting advertising Agency*	*A*	*2*
E	*Plan advertising cam-paign*	*D*	*4*
F	*Conduct advertising campaign*	*E*	*10*
G	*Design packaging of product*	–	*2*

NOTES

H	*Establish packaging facility*	*G*	*10*
I	*Package initial stocks*	*H, J*	*6*
J	*Order stock from manu-facturer*	*–*	*13*
K	*Select distributors*	*A*	*9*
L	*Sell to distributors*	*C, K*	*3*
M	*Transport stock to dis-tributors*	*I, L*	*5*

The logic of the predecessor activities for each activity listed in the above table should be understood properly. The project 'Launching a new product' can be broken down into a number of activities. The set of activities given in the table are one perception based on simple logic. Other such logic could also be developed. The students are advised to carefully study the precedence of the activities.

Solution. Network diagram for the activities listed in the table is shown in Figure 9.8.

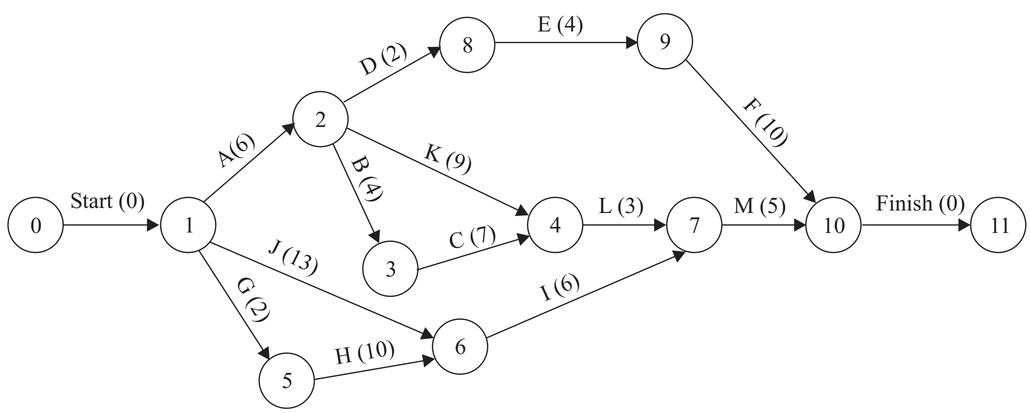


Fig. 12.8

Please see each activity carefully to understand the logic. The network is listed with 0 event and the activity has 0 time and is written as start (0). Arranging a sales office does not have any immediate predecessor activity. This is written as activity A with its time (6 weeks) written in the brackets as A (6) on top of the arrow. From node 1 there are three activities, which do not have any immediate predecessor activity, *i.e.,* A (6), G (2) and J (13). This may be verified from the precedence table. Activity B, hiring of salespersons can only commence after arranging sales office (activity A) so activity B (4) is shown as arrow coming out of node 2. Also activity D (2) and K (9) can also start only after activity A has been completed and they are shown with arrows moving out of node 2. There is only one activity C (7), which can start after completion of B and is shown as leaving node another node 11 has been created and the finish activity moving out of node 10. Finish activity has 0 times as shown in the network diagram.

NOTES

Example 12.2. *The characteristics of a project schedule are given below :*

S. No.	*Activity*	*Time*	*S. No.*	*Activity*	*Time*
1.	*1 – 2*	*6*	*2.*	*1– 3*	*4*
3.	*2 – 4*	*1*	*4.*	*3 – 4*	*2*
5.	3 – 5	5	6.	4 – 7	7
7.	5 6	8	8.	6 – 8	4
9.	8 – 7	2	10.	7 – 9	2
11.	8 – 9	1			

Construct a suitable network.

Solution. The network is shown in Figure 9.9

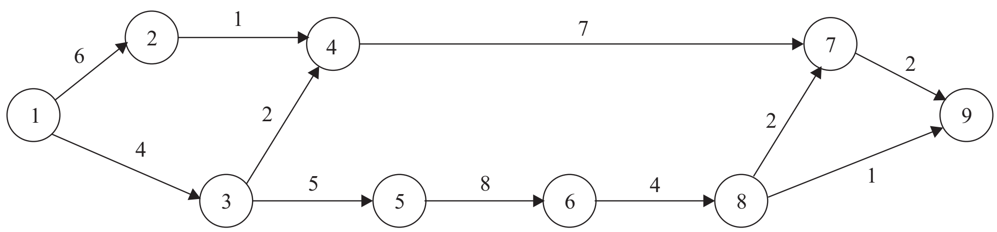


Fig. 12.9

Example 12.3. *Draw a network diagram based on the following project schedule infirmation available:*

S. No.	*Activity*	*Immediate Predecessor Activity*	*Time*
1.	*A*	*–*	*2*
2.	*B*	*–*	*4*
3.	*C*	*A*	*6*
4.	*D*	*B*	*5*
5	*E*	*C, D*	*8*
6	*F*	*E*	*3*
7	*G*	*F*	2

Solution. The network is shown in Figure 9.10.

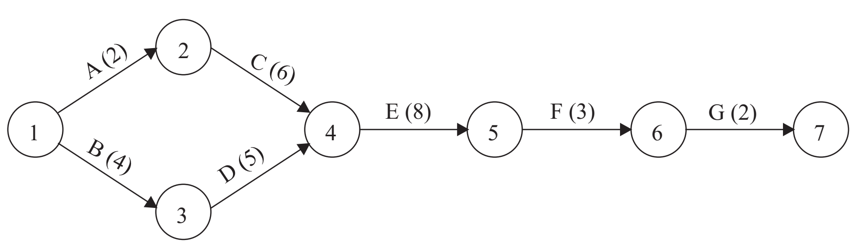


Fig. 12.10

Critical Path and Activity Times

NOTES

As explained earlier PERT is a very useful technique for planning the time and resources of any project. It is an event-oriented approach as it is mainly concerned with various events in a project. PERT deals with probability of completion of a project in particular time, as the time of various activities involved cannot be known accurately. It is only the time an activity is expected to take for completion, which can best be calculated. Expected time of completion of each activity can be found out form the following three timings :

(a) Optimistic Time

(b) The most likely time

(c) Pessimistic Time.

These three timings are based on Beta Statistical Distribution. Beta distribution is used as it is extremely flexible and can take on any form of activity and times that are associated in a typical project. Four typical Beta urves are shown below.

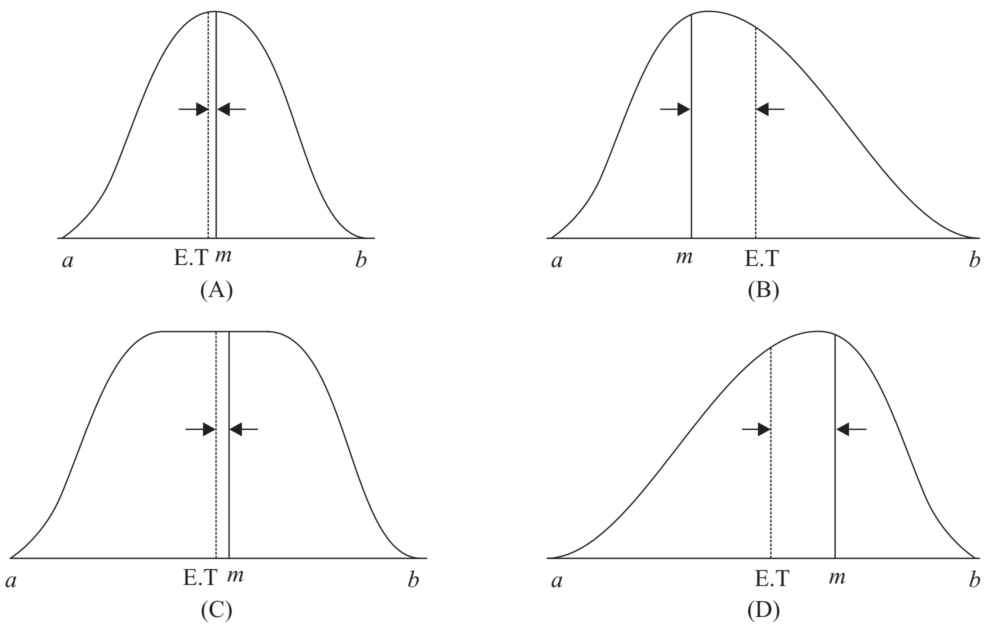


Fig. 12.11

It can be seen that the Beta distribution has finite end points like (*a*), the optimistic time and (*b*) the pessimistic time and the Expected Time (ET) of the activity is limited between these two ends. Curve (A) is a symmetrical curve and the difference between the most likely time (*m*) and Expected Time (ET) is very small. Had the curve been exactly symmetrical, the firm line (*m*) and dotted line (ET) would be exactly the same. Curve B indicates a high probability of finishing the activity *m* and ET indicates that if something goes wrong, the activity time can be greatly extended. Curve C is something like a rectangular distribution. Here the probability of finishing the activity early or late is almost equal. Similarly, curve D indicates very small probability of finishing the activity early but it is more probable that it will take an extended period of time. The Expected Time (ET) can be calculated from the following formula :

$$\text{ET} = \frac{a + 4m + b}{6}$$

Activity Times– Estimated Time

NOTES

After constructing a network reflecting the precedence relationship, we have to ascertain the time estimate for each activity. We must calculate ET for each activity using the above formula:

$$i.e., \qquad ET = \frac{a + 4m + b}{6}$$

Now the variance of the activity time has to be calculated.

$$V^2 = \left(\frac{b - a}{6}\right)^2$$

Earliest Start and Finish Times

Let us take zero as the start time for the project, then for each activity there is an Earliest Start Time (EST) relative to the project starting time. It is the earliest possible time that activity can start, assuming that all of the predecessor activities are also started at their EST. In that case for that particular activity, its Earliest Finish Time (EFT) is EST + activity time.

Latest Start and Finish Times

If we assume that the effort is to complete the project in as soon as possible time, this is the Latest Finish Time (LFT) of the finish activity or of the project. The Latest Start Time (LST) is the latest time when an activity can start, if the project schedule is to be maintained

LST = LFT – activity time

Finish activity has zero time, hence LST = LFT

Slack. Slack of an activity can be defined as the difference between the Latest Start Time (LST) and Earliest Start Time (EST) or the difference between the Latest Finish Time (LFT) and Earliest Finish Time (EFT). This is the significance of slack or Total Slack Time (TST), that the TST for any activity must be used up.

Critical Path

If we observe the network, we can see that there are a number of paths that lead to the finish activity, *i.e.,* completion of the project. But the longest path is the most limiting path. This path is called the *Critical Path*. It can be easily determined by adding the activity times of all the activities on the largest path from start to finish of the project.

Calculation of EST and EFT

These calculations can best be described with the help of a network. Let us draw a network as shown in Figure 15.14:

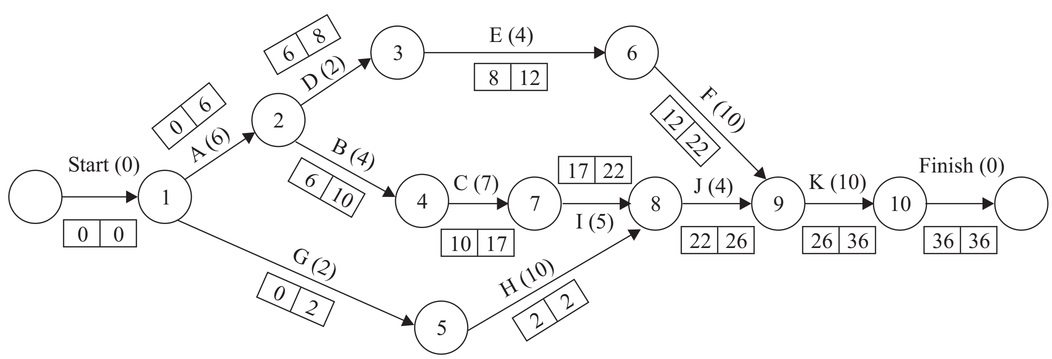


Fig. 12.12

This is the same network as was drawn in example 15.1

In the above figure the name of the activities are written above the arrow and their timings are written in the brackets. The start activity and the finish activity with zero timing have only been listed for convenience.

NOTES

For calculations of EST and EFT let us proceed forward through the network as follows :

(*a*) Put the value of the project start time in both EST and EFT positions near the start activity arrow. So for start activity EST and EFT is zero, which is placed under the start activity as [0 | 0].

(*b*) Consider activity A with activity time of 6. For this EST is zero and EFT is 6 because that is the minimum time the activity will take. It has been placed near activity A as [0 | 6].

(*c*) All activities emanating from node 2 will have EST as 6 and EFT = EST + activity tie, hence for activity B it is [6 | 10] because activity B has a timing of 4. Similarly, near activity D has been [6 | 8] as it has activity time of 2. All the timings have been written in this manner.

(*d*) Continue through the entire network and mark the EST and EFT. The critical path is ABCIJK and is 36. Hence for the finish activity EST = EFT = 36.

Calculation of LST and LFT

For this purpose we work backward through the network. These timings have been listed in Figure 9.13.

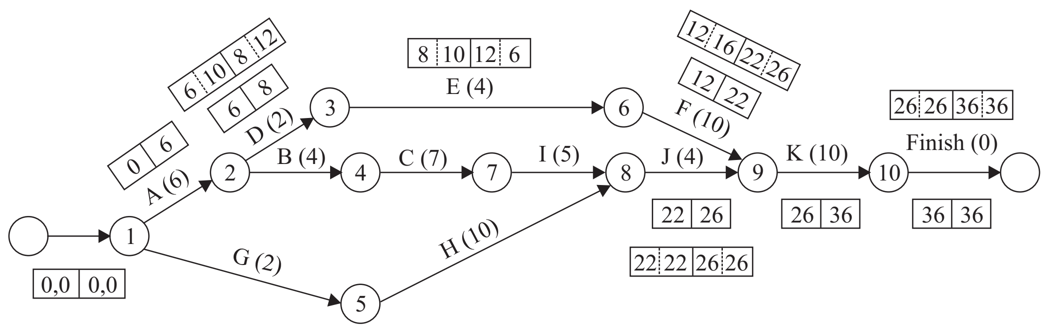


Fig. 12.13

For activity K, EST was 26 and EFT was 36. LST for this activity is 36 – 10 = 26 and LFT is 36 so mark it next to activity K as shown. Similarly, let us take activity F. EST was 12 and EFT 22 as activity time is 10. LST can only be 26 – 10 = 16 and LFT is 26.

Calculation of Float (Slack) and Crashing the Network

Example 12.4. *A project consists of the following activities. The Optimistic Time (OT), Pessimistic Time (PT) and Most Likely Time or the Expected Time for the activities is also listed in front of them.*

NOTES

Predecessor Activity	*Successor Activity*	*OT*	*Most Likely Time*	*PT*
1 – 2	*2*	*2*	*3*	*4*
2 – 3	*3*	*3*	*6*	*9*
2 – 4	*4*	*3*	*4*	5
3 – 5	5	2	4	6
3 – 6	6	–	0	–
4 – 6	6	–	0	–
4 – 7	7	4	5	6
5 – 7	7	4	6	8
6 – 7	7	6	7.5	12

Draw a network diagram of the above project and calculate associate timings of the project, i.e., Earliest and Latest Occurrence times of different events, slack, identify critical events and mark the Critical Path in the diagram. What is the total project duration ?

Solution. The network diagram is as shown below.

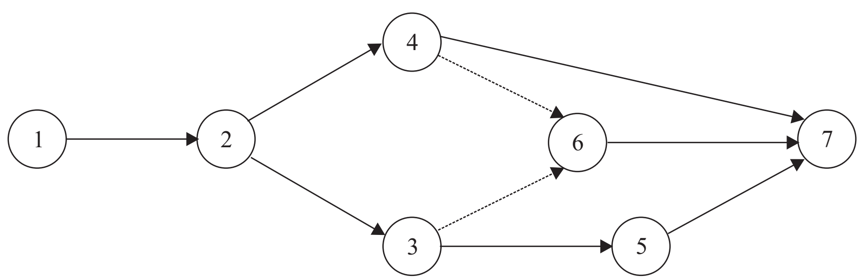


Fig. 12.14

(4, 6), (3, 6) → Dummy Activities

Critical Pt 1, –2, –3 –5 –7, Longest Path

Event

Predecessor	**Successor**	**ET =** $\frac{9 + 4m + 6}{6}$
1	2	3
2	3	6
2	4	4
3	5	4
3	6	0
4	6	0
4	7	5
5	7	6
6	7	8

NOTES

EST	LST
Event 1 =0	Event 7 =19
Event 2 =0 + 3 = 3	Event 6 =19 – 8 = 11
Event 3 =3 + 6 = 9	Event 5 =19 – 6 = 13
Event 4 =3 + 4	Event 4 =11 – 0 = 11
Event 5 =9 + 4 = 13	Event 3 =13 – 4 = 9
Event 6 =9 + 0	Event 2 =9 – 6 = 3
Event 7 =13 + 6 = 19	Event 1 =3 – 3 = 0

Now the slack can be calculated.

Event	**EST**	**LST**	**Slack**
1	0	0	0
2	3	3	0
3	9	9	0
4	7	11	4
5	13	13	0
6	9	11	2
7	19	19	0

All the events having zero slack are the Critical Events, *i.e.,* 1, 2, 3, 5 and 7. This is the Critical Path. The project duration is 19 (days/weeks).

Crashing of Network

Most of the projects result into cost overruns because of the inability of the project management team to complete the project in minimum possible time frame. The crashing of network involves considering the cost incurred on different activities required for completing the project. Let us understand certain terminology associated with crashing of network.

Normal Cost This is the cost of the project when all the normal activities are carried out, *i.e.,* there is no overtime or there are no special resources for which extra payment has to be made.

Normal Time It is that time in which project can be completed with the normal cost as defined above.

Crash Cost It is the minimum possible time, which is associated with the crash cost.

The relationship between these costs can be expressed as shown in Figure 9.15.

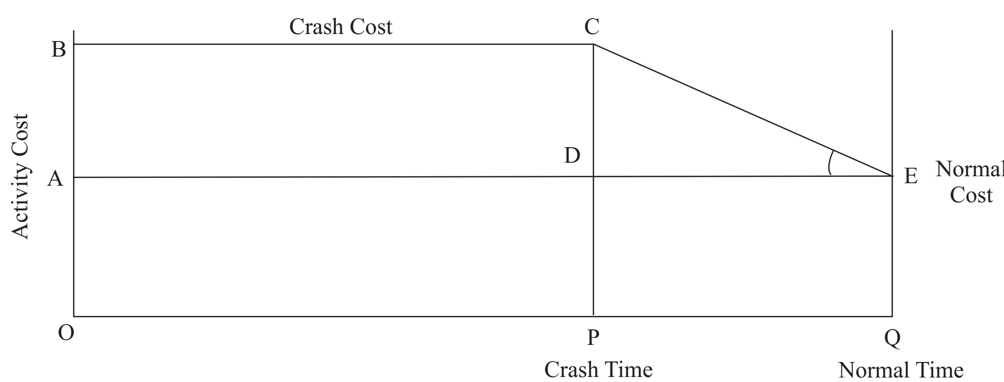


Fig. 12.15

It can be easily seen from the diagram above hat the cost-time slope (angle) is $\frac{OB - OA}{OQ - OP} = \frac{CD}{DE}$.

Slack

It should be appreciated that slack can refer to an activity as well as an event. It can be defined as the difference between the Latest Time and the Earliest Time. Since normally we deal with activity time in case of activities, slack and float have the same meaning. When slack is associated with an event, then the activity can have two slacks.

Head slack (slack of the head event) = LFT – EST of head event

Tail slack (slack of the tail event) = LFT – EST of tail event.

Float

Float can be described as the free time associated with an event. It is the time available for performing an activity in addition to the duration time. Hence, really float or slack is that time by which an activity can be delayed without delaying the entire project. These activities which do not have any, float or slack are the activities, which cannot be delayed without delaying the project. These activities are called the *critical activities*. Hence, along the critical path the float or slack is zero.

Float is an important concept in project planning. It helps the project management team to :

(*i*) priorities resources for allotment;

(*ii*) transfer of resources from one area to another depending upon where these are required earlier;

(*iii*) minimization of resources;

(*iv*) smoothen the use of resources.

Total Float

Total float is that time by which any activity can be maximum delayed without delaying the entire project. If the total float is used up in an activity, that particular activity and all the subsequent activities become critical.

Total Float = Latest occurrence time of the succeeding event–Earliest occurrence time of the preceding Event - duration of the activity.

Free Float

It is that time by which an activity can be delayed without effecting the commencement of a succeeding activity at its earliest start time. Free float results when all preceding activities occur at the earliest event times and all succeeding activities also occur at the earliest event times.

Free Float = Earliest occurrence time of the succeeding events – Earliest occurrence time of the Preceding events-duration of the activity.

Independent Float

Independent float is a measure of spare time that is available in an activity if it is started as late as possible and finished as early as possible. Hence, it is that amount of time by which an activity can be delayed, when all preceding activities are completed as late as possible and all succeeding activities are completed as early as possible.

Independent Float = Earliest occurrence time of the succeeding event–Latest occurrence time of Preceding event-duration of the activity.

Example 12.5. *A project consisting of eight activities is shown with the help of a network diagram below. Activity times have been marked on top of the arrow in brackets, calculate EST, LST, EFT and LFT. Also calculate the total float for each activity.*

NOTES

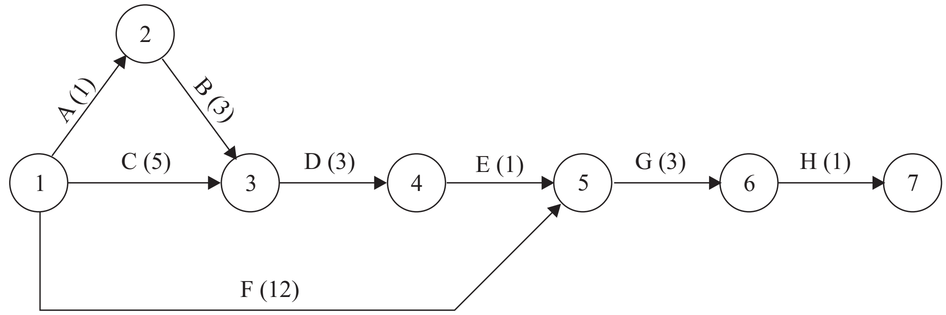


Fig. 12.16

Solution. EST = Earliest Event Time of the tail event

EET = EST (1 – 2) = 0

Earliest Event Time = Earliest occurrence time of the event preceding the event + duration = of the activity.

EST (1 – 3) = 0

EST (1 – 5) = 0

EST (1 – 5) = 0

EST (2 – 3) = 1

EST (3 – 4) = 5

EST (4 – 5) = 8

EST (5 – 6) = 12

EST (6 – 7) = 15

Also, let us calculate the Latest Finish Time from the above rework.

LFT (1 – 2) = 5

LFT (2 – 3) = 5 + 3 = 8

LFT (1 – 3) = 8

LFT (3 – 4) = 8 + 3 = 11

LFT (4 – 5) = 11 + 1 = 12

LFT (1 – 5) = 12

LFT (5 – 6) = 12 + 3 = 15

LFT (6 – 7) = 15 + 1 = 16.

LST can be calculated as

LST = LET – Duration of the activity converging on the head event

LST (1 – 2) = 5 – 1 = 4

LST (2 – 3) = 8 –3 = 5

LST (1 – 3) = 8 – 5 = 3

LST (3 – 4) = 11 – 3 = 8

LST (4 – 5) = 12 – 1 = 11

NOTES

LST (1 – 5) = 12 – 12 = 0

LST (5 – 6) = 15 – 3 = 12

LST (6 – 7) = 16 – 1 = 15

EFT can be calculated as follows :

EFT = EST + duration of the activity emanating from tail event

EFT (1 – 2) = 0 + 1 = 1

EFT (2 – 3) = 3 + 1 = 4

EFT (1 – 3) = 0 + 5 = 5

EFT (3 – 4) = 5 + 3 = 8

EFT (4 – 5) = 8 + 1 = 9

EFT (1 – 5) = 0 + 12 = 12

EFT (5 – 6) = 12 + 3 = 15

EFT (6 – 7) = 15 + 1 = 16

These timing can now be entered in the network.

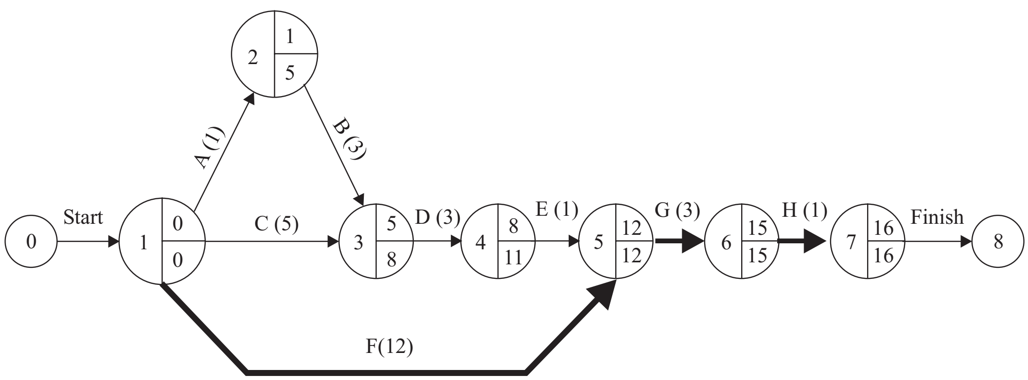


Fig. 12.17

EST have been entered on top half and LFT on the lower half.

Total Float can be calculated as follows :

Total Float = Latest occurrence time of the succeeding event – Earliest occurrence time of preceding event – duration of the activity.

= LST – EST

Float (1 – 2) = 4 – 0 = 4

Float (2 – 3) = 5 – 1 = 4

Float (1 – 3) = 3 – 0 = 3

Float (4 – 5) = 11 – 8 = 3

Float (1 – 5) = 0 – 0 = 0

Float (5 – 6) = 12 – 12 = 0

Float (6 – 7) = 15 – 15 = 0

These values of EST, LST, EFT and LFT as also the total float can be put in the form of a table as shown below.

NOTES

Activity	Duration	EST	LST	EFT	LFT	Total float
A	1	0	4	1	5	4
B	3	1	5	4	8	4
C	5	0	3	5	8	3
D	3	5	8	8	11	3
E	1	8	11	9	12	3
F	12	0	0	12	12	0 Critical
G	3	12	12	15	15	0 activities
H	1	15	15	16	16	0

Critical path FGH has been shown with thick line (1 – 5 – 6 – 7).

The total project duration = 12 + 3 + 1 = 16.

Project cost and crashing of activities

Project costs are the most vital aspects of project management, if due to any reasons, there are cost overruns, the entire decision making process may be affected adversely. One major advantage of Critical Path Method is that it is able to establish a relationship between time and cost. The management is always interested in cutting down the project time, since critical path measures the expected duration of the project time, through identification of the critical activities which need special attention. The aspect of project planning in which the project duration is intended to be reduced is called *project crashing*. It is desirable for the following reasons :

(*a*) Completing the project in the least possible time

(*b*) Reducing the project cost as far as possible

(*c*) Time and hence cost overruns can be minimized as the project managers can take measures to expedite other activities if the critical activities have taken more time than planned for.

(*d*) Reduction in idle time of the facilities and smoothing the utilization of the resources.

(*e*) Plans can be made to utilize the resources and facilities in efficient manner and these can be transferred /switched over to the other more profitable / desirable projects.

(*f*) The duration of the activities can be reduced by either allocating more resources in manpower and machines as originally planned for or by working over times in different shifts.

Example 12.6. *Draw a network from the following activities and find a critical path ad duration of the project.*

Activity	*Duration (days)*	*Activity*	*Duration (Days)*
1 – 2	*10*	*5 – 7*	*7*
2 – 3	*8*	*6 – 8*	*9*
3 – 4	*12*	*7 – 8*	*6*
3 – 5	*13*	*8 – 9*	*15*
4 – 6	*7*	*9 – 10*	*17*
5 – 6	*11*		

Solution.

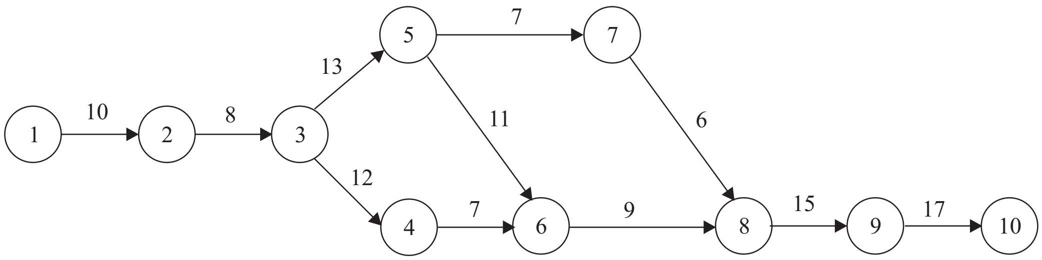


Fig. 12.18

Various paths	**Duration of paths**
1 – 2 – 3 – 5 – 7 – 8 – 9 – 10	10 + 8 + 13 + 7 + 6 + 15 + 17 = 76
1 – 2 – 3 – 5 – 6 – 8 – 9 – 10	10 + 8 + 13 + 11 + 9 + 15 + 17 = 83
1 – 2 – 3 – 4 – 6 – 8 –9 – 10	10 + 8 + 12 + 7 + 9 + 15 + 17 = 78

Hence the critical path is 1 – 2 – 3 – 5 – 6 –8 – 9 –10 with total duration of 82 day. It is marked with thick lines.

Example 12.7. *A small project consists of the following twelve jobs whose precedence relations are identified with their node numbers as follows :*

Job	**Precedence**	**Duration (Days)**	**Job**	**Precedence**	**Duration (Days)**
A	1 – 2	10	G	3 – 7	12
B	1 – 3	4	H	4 – 5	15
C	1 – 4	6	I	5 – 6	6
D	2 – 3	5	J	6 – 7	5
E	2 – 5	12	K	6 – 8	4
F	2 – 6	9	L	7 – 8	7

(*a*) *Draw a network diagram representing the project.*
(*b*) *Find the critical path and project duration.*
(*c*) *Calculate EST, EFT, LST, LFT for all the jobs.*
(*d*) *Tabulate Total Float, Free Float, Independent Float.*

Solution. The network diagram is shown below :

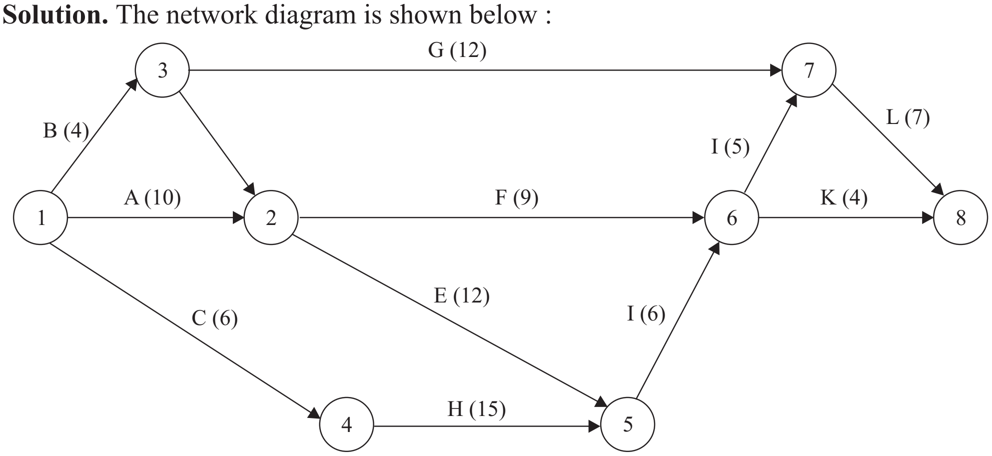


Fig. 12.19

NOTES

Paths	Duration
1 – 2 – 3 – 7 – 8	(10 + 5 + 12 + 7) = 34
1 – 2 – 6 – 7 – 8	10 + 9 + 5 + 7 = 31
1 – 2 – 6 – 8	10 + 9 + 4 = 23
1 – 2 – 5 – 6 – 7 –8	10 + 12 + 6 + 5 + 7 = 40
1 –2 – 5 – 6 – 8	10 + 12 + 6 + 4 = 32
1 – 3 – 7 – 8	4 + 12 + 7 = 23
1 – 4 – 5 – 6 – 7 – 8	6 + 15 + 6 + 5 + 7 = 39
1 – 4 – 5 – 6 – 8	6 + 15 + 6 + 4 = 31

Critical path is 1 – 2 – 5 – 6 – 7 – 8 with duration of 40 days. It is marked with thick lines in the network diagram.

Computation of EST, EFT, LST and LFT :

Job	**Duration**	**EST**	**LET**	**LFT**	**LST**	**Total Float**	**Head Event**	**Free Float**
(1)	(2)	(3)	(4)	5 = (3 + 2)	(6 = 4 – 2)	(7 = 6 – 3)	(8)	(9 = 7 – 8)
1 – 2	10	0	10	10	0	0	0	0
1 – 3	4	0	21	4	17	6	6	11
1 – 4	6	0	7	6	1	1	1	0
2 – 3	5	10	21	15	16	6	6	0
2 – 5	12	10	22	10	0	0	0	0
2 – 6	9	10	28	29	19	9	0	9
3 – 7	12	15	33	27	21	6	0	6
4 – 5	15	6	22	21	7	1	0	1
5 – 6	6	22	28	28	22	0	0	1
6 – 7	5	28	33	33	28	0	0	0
6 – 8	4	28	40	32	36	0	0	8
7 – 8	7	33	40	40	33	0	0	0

Terminology in Time Cost Relationship

(a) Normal Time of an activity (t_n)

(b) Crash Time of activity (t_c)

(c) Normal Cost (C_n)

(d) Crash cost of the activity (C_c)

(e) Activity cost slope or angle $= \dfrac{\Delta C}{\Delta T} = \dfrac{\text{Crash cost} - \text{Normal cost}}{\text{Normal time} - \text{Crash time}} = \dfrac{C_c - C_n}{t_n - t_c}$

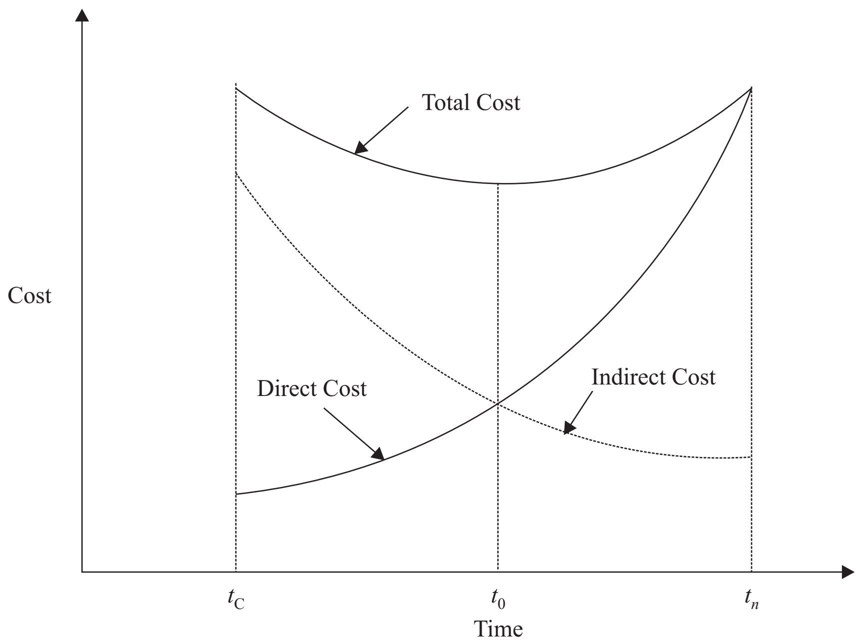


Fig. 12.20

It may be seen above, that the indirect cost of the project decreases with the increase in the duration of time, whereas direct cost increases with time and these two costs are opposite to each other. The sum of the two costs is shown as total cost. The project duration for which the total cost is minimum is called the *optimum time duration* shown as t_0.

12.4 STEPS IN PROJECT CRASHING

The following steps are involved in project crashing :

Step I. Calculation of the cost slope

As shown earlier cost slope = $\frac{\Delta C}{\Delta T} = \frac{C_c - C_n}{t_n - t_c}$

where $_c$, C_n, t_n and t_c have the usual meaning and $\frac{\Delta C}{\Delta T}$ denotes the cost of reducing duration of an activity by one unit of time.

Step II. Mark the critical path from which the expected duration of the project is found. Find the associated project cost for this critical path.

Step III. Select the least cost slope activity out of the critical path activities. If there happen to be more than one critical path, then select one such activity on each of the critical paths.

Step IV. Keep reducing the activity time of the selected activity unless and until either crash time is reached or the earlier non-critical parallel path becomes critical.

Step V. Step II to IV are repeated until we identify a critical path on which none of the activities can be further crashed.

Step VI. List the time and cost in the form of a matrix and select optimum duration of the project.

Example 12.17. *The network of a small project is shown below :*

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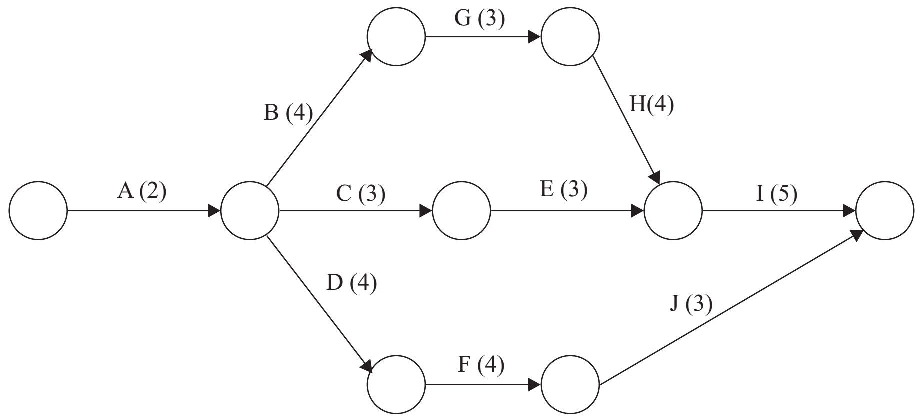

Fig. 12.21

The data for the cost and time is also given below. If the indirect cost of the project is estimated to be Rs.100 per day of the project duration what is the optimal project duration ?

Activity	Normal Time (days)	Crash Time (days)	Normal Cost (Rs.)	Crash Cost (Rs.)
A	2	1	70	80
B	4	2	80	200
C	3	1	130	230
D	4	2	130	300
E	3	3	120	120
F	4	2	80	120
H	4	2	100	280
I	5	2	80	200
J	3	2	60	00

Total Normal Cost = 1120

Solution. Step I. Calculation of the cost slopes of the each of the activities of the project.

$$A = \frac{80-60}{2-1} = 20$$

$$B = \frac{200-80}{4-2} = 60$$

$$C = \frac{100}{2} = 50$$

$$D = \frac{100}{2} = 50$$

$$E = 0$$

$$F = \frac{40}{2} = 20$$

$$G = \frac{180}{2} = 90$$

$$H = \frac{140}{2} = 70$$

$$I = \frac{120}{3} = 40$$

$$J = .\frac{30}{1} = 30$$

Step II. Identify critical path and find the expected duration of the project and direct cost of the project.

CP = A –B –G – H – I, Expected normal duration = 2 + 4 + 3 + 4 + 5 = 18 days

Direct Cost = Rs. 1120

Step III. Least cost activity on the critical path is A, as it has the lowest cost slope of 20 and this can be crashed by 1 day (crash time = 1 day given in the problem, *i.e.,* 2 – 1 = 1)

New duration = 18 – 1 = 17 day

New cost = Rs. 1120 + 1 × 20 = Rs. 1140

The new network with activity A crashed (circled to show that it has been crashed) it shown below.

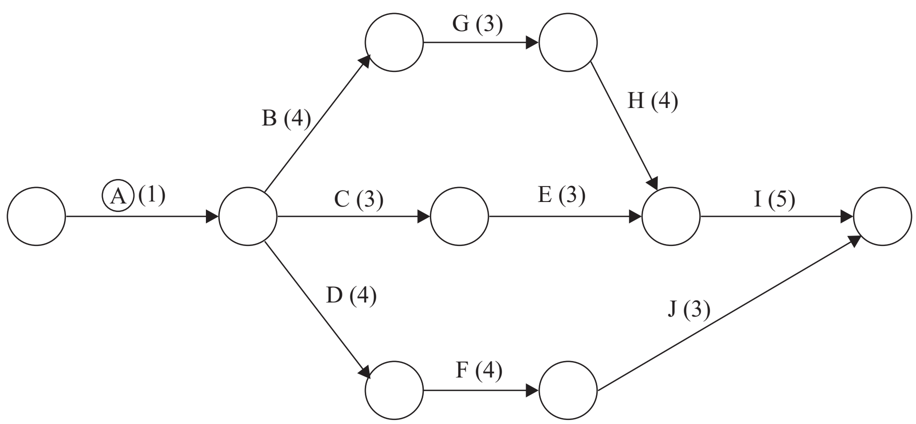

Fig. 12.22

Step IV. Repeat step II to III

Out of the remaining activities on critical path, BGHI, activity I has the lowest unit cost of crashing of 40. It can be crashed by (5 – 2) = 3 days.

New Duration of the project = 17 – 3 = 14 days

New Project cost = Rs. 1140 + 3 × 40

= Rs. 1260

Out of the remaining three activities on the CP, i.e., BGH activity B has the lowest cost of 60 and it can be crashed by = 4 – 2 = 2 days

New project duration = 14 –2 = 12 days

New Project Cost = 1260 + 2 × 60 = Rs. 1380

The new network ay be drawn to show the impact of crashing.

NOTES

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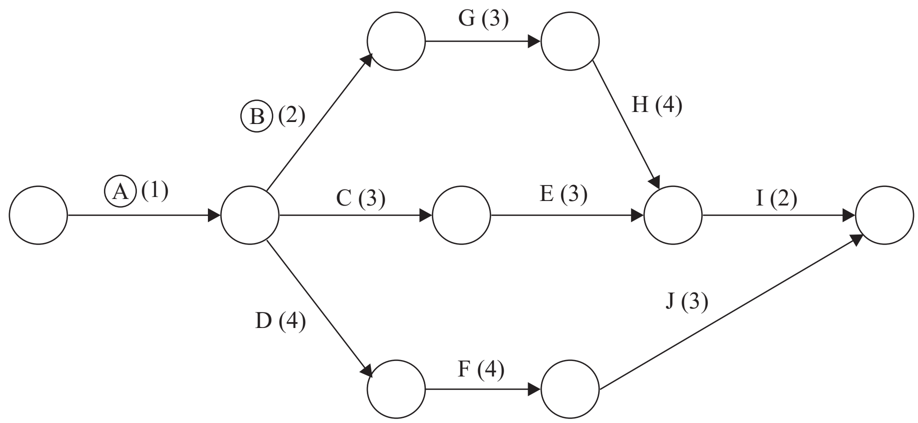

Fig. 12.23

With crashing of B, the path ADFJ has also become critical. To crash the project duration further, we select one activity from each of the two critical paths and crash each selected activity by smallest of duration by which these activities can be crashed.

On path ABGHI, G and H are left for crashing and in the path ADFJ, three activities DFJ can be crashed. Since both activities H and F can be crashed by 2 days (*i.e.,* H = 4 – 2 = 2, F = 4 – 2 = 2), it will results in

New Project duration = 12 – 2 = 10

Project Cost = $1380 + 2 \times 70 + 2 \times 20$ as cost slope of Hand F are Rs. 70 and 20 respectively

= Rs. 1560

The crashed network is shown below.

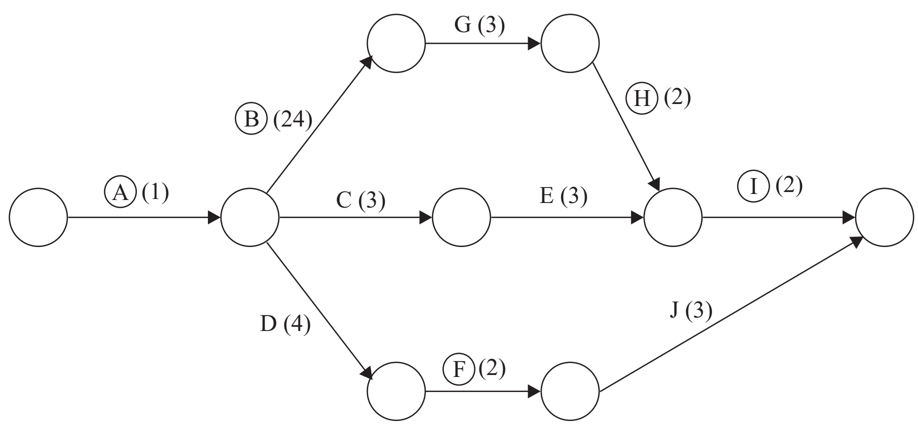

Fig. 12.24

On the original critical path only one activity G remains to be crashed which can be crashed by 2 days but costs Rs. 90/ to crash per day. On the other CP activities D and J remain to be crashed which cost Rs. 50 and Rs. 30 to crash per day. Since their total cost is less than Rs. 90 (*i.e.,* Rs. 50 + Rs. 30 < Rs. 90) activities D and J have been selected to be crashed. D can be crashed by 2 days but J can be crashed by one day, hence both will be crashed by one day.

New Project duration = 10 – 1 = 9 days

New Project cost = $1560 + 90 \times 1 + 30 \times 1$

= Rs. 1680

It can be seen in the network drawn below that the crashing of activities G and J have made all the three paths critical. Now only one activity *i.e.,* G remains to be crashed on CP, A – B – G – H – I, similarly only activity D remains to be crashed on CP A – D – F – J. But on the third CP, ACEI two activities C and E remain to be crashed. We have to select one activity each from each of the CPs and crash it. From CP, A – C – E – I activity C will crash since E cannot be crashed technically. So, activities G, C and D have to crash. Out of these G could originally be crashed by 2 days but it has already been crashed by one day. All the three activities on the three CPs will be crashed by one day.

New Project duration = 9 – 1 = 8 days

New Project cost = 1680 + 90 × 1 + 50 × 1 + 50 × 1

= Rs. 1870

The finally crashed network can be shown below.

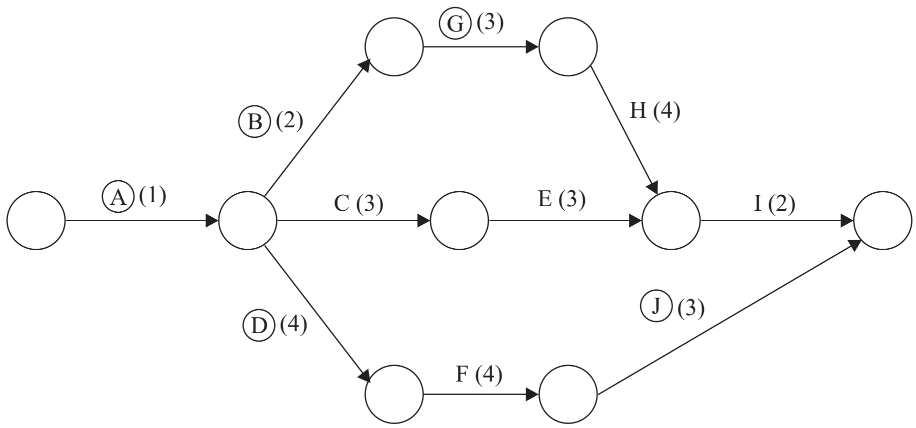

Fig. 12.25

Crashed duration of the project = 8 days on all three critical paths the total duration is 8 days only.

Step V. List the project-time cost in a table and select the optimal duration of the project.

These are drawn in the following table.

Project Duration (days)	**Direct cost (Rs.)**	**Indirect cost (Rs.) @ Rs. 100 per day**	**Total Project Cost.**
18	1120	1800	2920
17	1140	1700	2840
14	1260	1400	2660
12	1380	1200	2580
10	1560	1000	2560
9	1680	900	2580
8	1870	800	2670

It may seen that the cost is minimum when the project duration is 10 days. The result of crashing exercise undertaken above can be summarized as

NOTES

Normal duration of the project = 18 days

Crashed duration of the project = 8 days

Optimal duration of the project = 10 days

Minimum cost of the project = Rs. 2560

12.5 PROBABILITY AND PROJECT PLANNING

As explained earlier in this chapter, PERT is able to provide help in decision-making under conditions of uncertainty. Uncertainty is almost always associated with the project completion time and completion of different activities in planned time.

Using the concept of time estimates, optimistic time, most likely time and pessimistic time and the formula associated with these,

$$T_{c_p} = \text{Expected time of completion of the project}$$

$$= \sum t_{e_1} + t_{e_2} + t_{e_3} + ... + t_{e_n}$$

where are the expected times of the activities on critical path and $V_1, V_2, ..., V_n$ are te variances of the activities.

$$\text{Variance V} = \frac{b-a}{6} \text{ and Standard Deviation s} = \left(\frac{b-a}{6}\right)^2$$

then $$\sigma = \sqrt{V_1 + V_2 + V_3 + ... + V_n}$$

Example 12.9. *Activities of a small pr re given below. The network of this project is also drawn. What is the probability of completing he project within 26 days, within 28 days.*

Activity	Most optimistic time (days)	Most likely time (days)	Most pessimistic time (days)
1 – 2	1	1	1
2 – 3	1	4	7
2 – 4	8	12	10
3 – 5	3	5	7
4 – 5	1	1	1
5 – 6	3	6	9
5 – 7	4	6	8
6 – 8	4	8	12
7 – 8	2	5	8

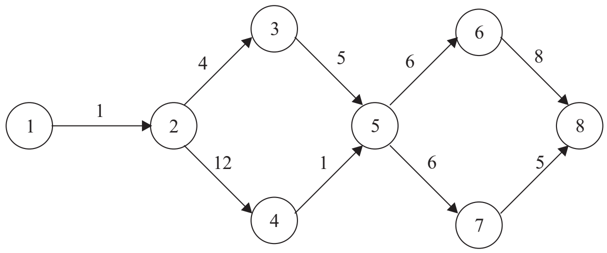

Fig. 12.26

Solution. The expected time t_e and the variance of different activities can be fond out. It is given in the following table :

EST for each activity can be calculated

Node 1 = 0

Node 2 = 0 + 8 = 8

Node 3 = 8 + 4 = 12

Node 4 = 8 + 12 = 20

Node 5 = Maximum out of [(12 + 5), (20 +1)] = 21

Node 6 = 21+ 6 = 27

Node 7 = 21 + 6 = 27

Node 8 = Maximum out of [(21 + 8) and (21 + 5)] = 29.

Similarly, LST for each activity can be calculated.

LST Node 8 = 29

Node 7 = 29 – 5

= 24

Node 6 = 29 – 8 = 21

Node 5 = Maximum out of [(21 – 6), (24 – 6)]

= 15

Node 4 = 15 – 3

= 12

Node 3 = 15 – 5 = 10

Node 2 = Min out of [(12 – 11), (10 – 4)]

= 1

Node 1 = 1 – 1 = 0

Let us redraw the network showing the critical path with a thick line.

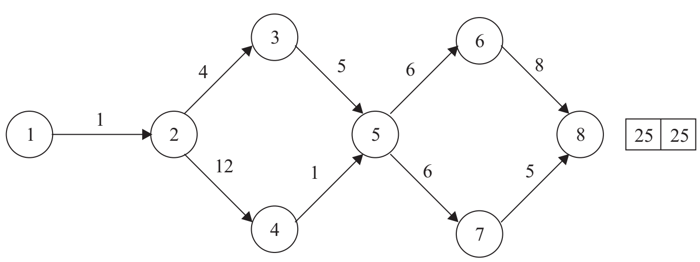

Fig. 12.27

Critical path is 1 – 2 – 4 – 5 – 7 – 8.

Expected time in completing the project = 1 + 12 + 1 + 6 + 25 days

NOTES

$$\text{Project variance} = \sigma^2 = 0 + 0.111 + .027 + 0.44 + 1$$

$$= \mathbf{1.578}$$

$$\sigma = .\sqrt{1.578} = 1.256$$

Probability of completing the project in 30 days,

$$Z = \frac{X - \overline{X}}{\sigma} = \frac{26 - 25}{1.256} = 0.796$$

where, X = 30 days (time under consideration)

$\overline{X}$ = Length of critical path = 25 days

σ = SD of critical path

The value from the cumulative normal distribution table for Z = 0.796 is 0.7852. Hence, the probability of completing the project within 28 days is 79.6 %.

Similarly, when we have to find probability of completing the project in 28 days,

$$Z = \frac{28 - 25}{1.256} = 2.388$$

The value from the table for Z = 2.388 is 991576

i.e., the probability of completing the project in 28 days is 99.15%

Example 12.10. *An R & D project has large number of activities but the management is interested in controlling a part of these activities 7, in number. The followig date is available for these 7 activities :*

Activity	Preceding activity	(a)	Times (m)	(b)
A	None	4	6	8
B	A	6	10	8
C	A	8	18	10
D	B	9	9	9
E	C	10	4	4
F	A	5	5	5
G	D, E, F	8	6	10

(i) *Draw a PERT network for the activities shown in the table.*

(ii) *Prepare the schedule of the 7 activities.*

(iii) *Mark the critical path on the network.*

(iv) *If the management puts a deadline of 37 days for completion of this part of the project, determine the probability it will be completed in 37 days.*

(v) *When should the management start these activities to get a confidence level of 99% of completion of these activities in the scheduled time* ?

NOTES

Solution. The network for the above data an be drawn as shown in Figure 9.28 :

(i)

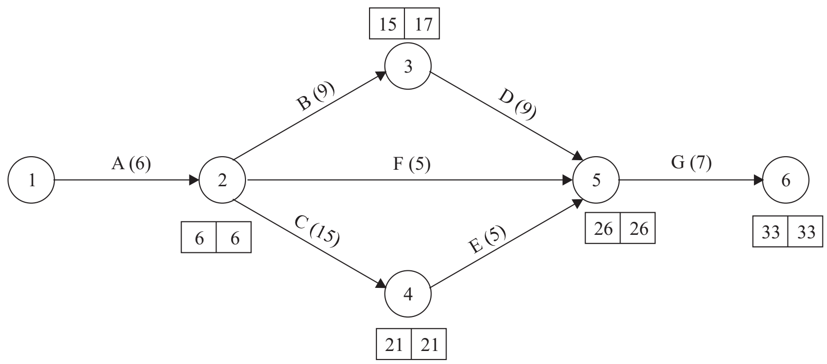

Fig. 12.28

(ii) Time for each activity has been determined using the formula

$$t = \frac{a + 4m + b}{6}$$ and shown is brackets along with the activities.

Activity	A	B	C	D	E	F	G
Time	6	9	15	9	5	5	7

The critical path is A – C – E – G marked with thick lines and the expected length of this part of project = 6 + 15 + 5 + 7 = 33 days.

Now let us work out the variance, i.e., σ^2 using the formula $\left(\frac{b-a}{6}\right)^2$

Activity	A	B	C	D	E	F	G
Time	0.444	0.111	0.111	0	1	0	0.111

$$\text{Variance} = 0.444 + 0.111 + 0.111$$

$$= 1.666 \text{ or } \sigma = \sqrt{1.666} = 1.29$$

EST and LST have been shown along side each node.

(*iii*) Probability that te project will be completed in 37 days.

$$Z = \frac{37-39}{1.29} = \frac{-2}{1.29} = -1.55$$

For Z = – 1.55 the value from the tables is 0.93943.

i.e., the probability that this part of the project will be completed in 37 days is 93.94 %.

(*iv*) For 99% assurnce the Z value from the table is 2.33.

$$Z = \frac{X-39}{1.29}$$ we can substitute Z value in this.

$$2.33 = \frac{X-39}{1.29} \text{ or } X - 39 = 2.33 \times 1.29 = 3$$

or X = 42

The management has 99% assurance, that this part of project will be completed in 42 days.

NOTES

Example 12.26. *Given below is the list of activities along with their predecessor activties. Three time estimates are also provided.*

Activity	Predecessor Activity	Most optimistic (a)	Time (weeks) Most likely (m)	Most pessimistic (b)
A	NIL	1	2	9
B	A	2	3	4
C	A	2	4	6
D	A	3	5	7
E	C	5	7	9
F	D	1	3	5
G	B	1	4	7
H	G	2	6	10
I	E, H	4	8	6
J	F	2	6	10

What is the probability of critical path being completed in (i) 23 days (ii) 21 days ?

Solution. For drawing the network we need the activity time, which can be calculated using the relationship . Also for finding out the probabilities the σ^2 must be calculated. These calculations ae given below.

Now the network can be drawn

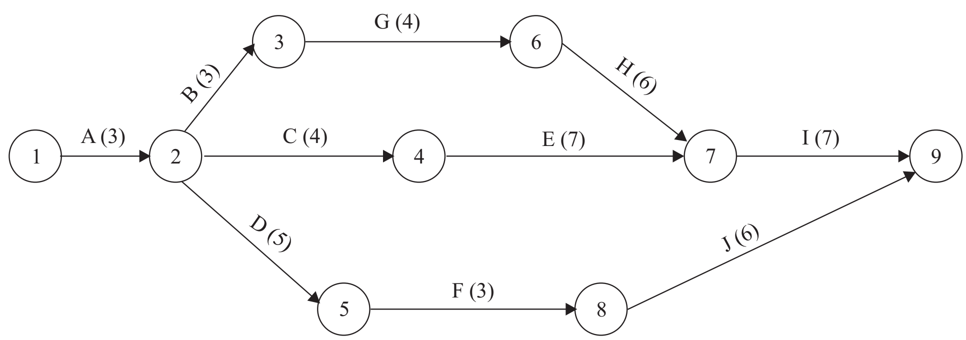

Fig. 12.29

The activity timings have been shown in brackets along with the activity on top of the arrow.

The critical path, the path with longest duration is ABGHI and the total duration of the activities on critical path is 3 + 3 + 4 + 6 + 7 = 23 weeks. It is marked with thick lines in the network.

$$\text{Variance } \sigma^2 \text{on critical path} = 3.16 + 0.11 + 1 + 1.77 + 0.11$$

$$\sigma^2 \;\; 6.15$$

$$\sigma = 2.48$$

$$\text{Standard normal deviation } Z = \frac{\text{Scheduled time} - \text{Duration of the critical path}}{\text{Standard Deviation of critical path } (\sigma)}$$

If the scheduled time of completion is 25 days as given in the problem,

$$\text{then} \qquad Z = \frac{25 - 23}{2.48} = 0.8051$$

Hence, probability of completion of the critical path of the project is 80.51%. If the scheduled time is 21 days.

$Z = \frac{12 - 23}{2.48} = -0.806$. Ignoring the negative value read the probability value, which is 0.8051.

∴ Probability of completing the project is 21 days = 1 – 0.8051 = 0.195

i.e., the probability completing the project in 21 days 1.95%

REVIEW AND DISCUSSION QUESTIONS

1. What is the critical path analysis? What are the areas where this technique can be applied?

2. How does PERT differ form CPM ? Describe briefly the basic steps to be followed in developing PERT/CPM programmed ?

3. Under what circumstances would you use PERT as opposed to CPM is project management ? Name a few projects where each would be more suitable than other.

4. What is the significance of three times estimates used in PERT ? How an on what basis is a single estimate derived from these estimates ?

5. What is critical path ? What does it signify ? What are its benefits ?

6. Describe with the help of a diagram, the procedure to arrive at the critical path in a PERT network.

7. What do you understand from earliest finish time and latest finish time ? How are they calculated ? Explain your answer with an example.

8. What do you understand from slack ? What are the different types of slacks ? How does knowledge of slack help better project management ?

9. A management institute plans to organize a conference on the use of "Quantities techniques for decision making". In order to coordinate the project, it has decided to use a PERT network. The major activities and times estimate for ach activity have been completed as follows :

Activity Description	Times Estimate	Activity that must precede
a. Design conference meeting theme	1-2-3	None
b. Design front cover of conference proceedings	1-2-3	A
c. Design brochure	1-2-3	A
d. Compile list of distinguished speakers	2-4-6	A
e. Finalize brochure and print it	2-5-14	C and D
f. Make travel arrangements for distinguished speakers	1-2-3	D
g. Send broachers	1-3-5	E
h. Receive papers for conference	10-12-30	G
i. Edit papers	3-5-7	H
j. Print proceedings	5-10-15	B and I

(a) Construct an arrow diagram called network.

(b) Calculate expected time for each activity.

(c) Identify critical path and determine the project duration.

NOTES

10. Major activities involved in the develpment of an item with a vendor are as under:

Activity	Duration	(Weeks)
A	...	2
B	...	1
C	...	2
D	...	1
E	...	5
F	...	8
G	...	4
H	...	2
I	...	1
J	...	4

Constraints:

(i) A is start activity.

(ii) B can start on completion of A.

(iii) C, E and H succeed B

(iv) C controls D, E controls F and H controls I

(v) G can commence after F is over.

(vi) J can start once D and I are over.

(vii) G and J are last activities.

(a) Draw the project network and identify all the paths.

(b) How many weeks are required by the vendor to develop the item ?

(c) What suggestions do you make to reduce the development time ?

11. A company manufacturing plant and equipment plant for chemical processing is in the process of quoting a tender called by a Public Sector Undertaking. Delivery date once promised is crucial as penalty clause is applicable. The wining of tender also depends on how soon the company is able to deliver the goods. Project manager has listeddown the activities in the project as under :

S. No.	Activity	Immediate proceeding activity	Activity Time (weeks)
1	A	...	3
2	B	...	4
3	C	A	5
4	D	A	6
5	E	C	7
6	F	D	8
7	G	B	9
8	H	E, F, G	3

(a) Find out the delivery week from the date of acceptance of quotation.

(b) Find out total float and free float for each of the activities.

12. Calculate EST, EFT and LFT for the following network. The duration for each activity is given on upper side of arrow line.

13. Time and cost data of the ctivities of a small project is given below :

Activity	Normal		Crash		Cost slope	
	Time (Days)	Cost (Rs.)	Time (Days)	Cost (Rs.)	Time (Days)	Cost (Rs.)
1 – 2	3	360	2	400	1	40
2 – 3	6	1,440	4	1,620	1	90
2 – 4	9	2,160	5	2,380	4	55
2 – 5	7	1,120	5	1,600	2	240
3 – 4	8	400	4	800	4	100
4 – 5	5	1,600	3	1,770	2	85
5 – 6	8	480	7	769	1	280

The overhead cost per day is Rs. 160.

(i) Find critical path.

(ii) Crash the project to achieve optimum duration and optimum cost.

14. A project consists of nine activities. Activities are identified by their beginning (*i*) and ending (*i*) node numbers. The three estimates are listed in the table below.

Activity (*i* – *j*)	Estimated duration (weeks)		
	Optimistic	Most likely	Pessimistic
1 – 2	1	1	7
1 – 3	1	4	19
1 – 4	1	4	7
4 – 5	2	5	14
2 – 6	2	5	8
5 – 6	1	4	19
5 – 6	1	4	19
3 – 7	2	5	14
6 – 7	3	6	15

(a) Draw the project network and identify all the paths through it.

(b) Identify the critical path and determine the expected project duration.

(c) Calculate variance and standard deviation of the project duration.

(d) What is the probability that the project will be completed.

(i) At least 2 weeks earlier than expected ?

(ii) Not more than 2 weeks later than expected ?

(e) What due date has a probability of completion f 0.95 ?

NOTES

Given normal distribution function

Normal Deviate (z)	Probability %	Normal Deviate (z)	Probability %
– 0.9	18.4	+ 0.9	81.6
– 0.1	15.9	+ 1.0	84.1
– 1.1	13.6	+ 1 .1	86.4
–1.2	11.5	+ 1.2	88.5
– 1.3	9.7	+ 1.3	90.3
–1.4	8.1	+1.4	91.3

15. The following table gives for each activity of a project, its duration and responding resource requirement as well as otal availability of each type of resources:

Activity	Duration (Days)	Resources (machines)	Required (men)
1–2	7	2	20
1–3	7	2	20
2–3	8	3	30
2–4	6	4	20
3–6	9	2	20
4–5	3	2	20
5–6	5	4	40

Minimum available Resources.

(i) Draw the Network, compute earliest Occurrence Time and Latest Occurrence Time for each event, the total float each activity and identify the critical path assuming that there are no resource constraints.

(ii) Under the given resource constrains find out the minimum duration to complete the project and compare the utilization of the resources for the duration.

16. A projection consists of 10 activities, each of which requires either, or both, of the two types of resources R_1 and R_2 for its performance. The duration of the activities an their resource requirements are as follows :

Activity	Duration (days)	R_1	R_2
1–2	3	3	2
1–3	2	6	–
1– 4	6	4	–
2–6	4	–	4
3–5	2	2	2
4–5	1	4	–
4–8	4	4	–
5–7	3	3	2
6–7	2	1	3
7–8	4	4	5

NOTES

Resource availability : 8 units of R_1 and 5 units of R_2

Determine the duration of the project under given resource constraint. If the resources were not a problem, how long would the project take to complete in the normal course ?

17. Explain the meaning of 'crashing' in network techniques.

18. What do you understand by the term direct cost and indirect cost in PERT costing techniques ? How do they behave in project cost with range of duration ?

19. (a) What do you mean crash duration ?

(b) Write a short note on project crashing using network analysis. (Also give graph for cost slope).

20. What is a least-cost schedule of a project ? How is it obtained ?

21. How do you distinguish between resource levelling and resource allocation problems ? State and explain an algorithm of resource allocation.

22. Explain how network analysis can be used for resource planning and levelling in project management.

23. Explain the use of float in levelling of resources.

24. Give a procedure of resource levelling using PERT/CPM.

25. Distinguish between 'Precedence Diagram' and 'Network Diagram'.

26. A small project consisting of 8 ctivities has the following characteristics :

Activity	**Preceding Activity**	**Time Estimate Weeks)**		
		Most Optimistic	**Most likely**	**Most Pessimistic**
A	None	10	5,000	7,000
B	A	8	4,300	5,900
C	A	4	3,700	4,900
D	C	3	2,500	2,500
E	B	6	3,900	4,800
F	D	5	4,600	6,100
G	D,E	4	2,800	2,800
H	F	3	4,100	6,200
I	G	5	4,800	7,800

(a) Draw the PERT network for the project.

(b) Determine the critical path.

(c) If a 30 weeks deadline is imposed, what is the probability that the project will be finished within the time limit ?

NOTES

27. The following information relates to a construction project for which your company is about to sign a contract. Seven activities are necessary and the normal duration, normal cost, crash duration and crash cost have been derived from the best available sources.

Activity	Preceding Activity	Duration in Weeks		DirectQ Cost (Rs.)	
		Normal	Crash	Normal	Crash
a	–	15	12	4,500	5,250
b	–	19	14	4,000	4,500
c	–	9	5	2,500	4,500
d	A	6	5	1,700	1,940
e	A	14	9	4,300	5,350
f	b, b	9	6	2,600	3,440
g	e	8	3	1,800	3,400

Each activity may be reduced to the crash duration in weekly stages at pro rata cost. There is a fixed cost of Rs. 500 per week.

Required :

(a) Draw, clearly labelled, a network and indicate the notation pattern used.

(b) Indicate the critical path and state the normal duration and cost.

(c) Calculate the critical total cost, showing clearly four working, and the vised duration and cost for each activity